PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

On the Application of the Theory of Error to Cases of Normal Distribution and Normal Correlation

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Phil. Trans. R. Soc. Lond. A 1899 **192**, 101-531
doi: 10.1098/rsta.1899.0003

III. *On the Application of the Theory of Error to Cases of Normal Distribution and Normal Correlation.*

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Received (under the title "On the Geometrical Treatment of the 'Normal Curve' of Statistics," &c.) October 9, 1897,—Read November 25, 1897,—Revised July 15, 1898.

Contents.

15.12.98

Introductory.

In his 'Lettres sur la Théorie des Probabilités' (1846), QUETELET has shown that in certain anthropometrical statistics, *e.g.*, in statistics of height or of chest-measurement, the curve of frequency is approximately of the same form as the curve known to mathematicians as the "curve of error," but better described for statistical purposes as the *normal curve.* A similar conclusion has been arrived at by later observers with regard to a large number of biological measurements. The general similarity thus established has been extended, primarily by Mr. FRANCIS GALTON, to certain cases of statistical correlation of two or more attributes. It has been found in these cases that not only are the curves of frequency of the separate attributes approximately normal curves, but the frequencies of joint occurrence of different measures of these attributes follow (approximately) a simple law, corresponding to the law of correlation of errors of observation.

Since we can never observe more than a finite number of individuals, it is impossible to decide with absolute certainty as to the existence, in any particular case, of this (or any other) law of distribution or correlation. But if the number of observed individuals is large, and if they are obtained by random selection from a "community" comprising (practically) an indefinitely great number of individuals, the theory of error provides us with a test for deciding whether any particular law, suggested by the given observations, may be regarded as holding for the original community.

The main object of the present memoir is to obtain formulæ for testing the existence, in any particular case, of the *normal distribution* and *normal correlation* described above. As the treatment of multiple correlation presents some difficulty, I have restricted myself to the cases of one attribute, supposed to be normally distributed, and of two attributes, supposed to be normally correlated. Where the hypothesis of normal distribution or of normal correlation may be regarded as established, there are different methods of treating the statistical data; and these may lead to different results. I have therefore given formulæ for comparing the relative accuracy of different methods of calculating the frequency-constants which are required.

The application of the formulæ to actual cases is postponed until certain tables are completed. In the absence of these tables, KRAMP's and ENCKE's tables (printed at the end of DE MORGAN's article on the "Theory of Probabilities" in the 'Encyclopædia Metropolitana') may be used for cases of a single attribute. For cases of correlated attributes, I have given two methods of making a rough calculation of the "theoretical" distribution, for comparison with the "observed" distribution. These methods depend on theorems which can be conveniently expressed in a geometrical form. As the normal curve lends itself to geometrical treatment, and as the fundamental formulæ in the theory of error can be obtained by the use of ordinary algebra, I have attempted to make the memoir complete in itself by starting with a simple definition of the normal curve, and adopting GALTON's definition of normal correlation; and by deducing the necessary theorems without the direct use of the differential or integral calculus.

The normal curve may be defined in various ways, *e.g.* :—

(1.) *Functional Equation*, $z = f(x^2)$, where $f(x^2) \times f(y^2) = f(x^2 + y^2)$.

(2.) *Ordinary Cartesian Equation*, $z \propto e^{-\frac{1}{2}(x^2/a^2)}$.

(3.) *Differential Equation*, $a^2(dz/dx) + xz = 0$.

(4.) *Geometrical Equation*, abscissa $\times$ sub-tangent $=$ constant. This follows at once from (3); for if O is the foot of the central ordinate, and if MP is any other ordinate, and the tangent at P meets OM in T, then sub-tangent MT $= -z\,dx/dz$.

(5.) *Statistical Equation*, $\lambda_{k+2} = (k+1)\lambda_2\lambda_k$, where λ_k denotes the mean kth power of the deviation from the mean in a distribution whose curve of frequency is a normal curve; k being any positive integer. This relation follows from (3). Since, by the definition, $\lambda_1 = 0$, it gives λ_k in terms of λ_2 for all positive integral values of k; and it may therefore be regarded as the equation to the curve, the position of the central ordinate being arbitrary.

Of these different equations the first is in some respects the most important, as it is the direct expression of the relation on which the special property of normal distributions depends; the property, that is to say, that if the measures of a number of independent attributes are normally distributed, any linear function of these measures is also normally distributed. The second equation is, of course, essential for any numerical calculations. The last two, however, have certain conveniences when an elementary investigation is desired. I have therefore adopted the *geometrical definition* of the curve, and have deduced the statistical equation; and then have used either or both of these as occasion might require.

The memoir is divided into four parts. Part I. deals with elementary theorems; most of these are well known, but it is convenient to have them collected, and established by comparatively simple methods.* Part II. contains the investigation of the principal formulæ in the theory of error as applied to numerical statistics. In Part III. these formulæ are applied to cases of normal distribution. Part IV. deals with normal correlation, and is subdivided into two portions. The first consists of a discussion of the more important phenomena which occur when two attributes are normally correlated; while the second contains the applications of the theory of error. Some of the formulæ given in Parts III. and IV. have already been obtained by Professor KARL PEARSON, but by a different method.

PART I.—GENERAL PROPERTIES OF THE NORMAL CURVE AND OF NORMAL DISTRIBUTIONS.

The Normal Curve.

§ 1. *Definition of Normal Curve.*—Let O be a fixed point in a straight line X′OX, and let a point P move so that, if MP is the ordinate to P from X′OX, and PT the tangent at P, intersecting X′OX in T, the rectangle OM.MT is constant and $= a^2$. Then the path of P is a *normal curve.*

Let OZ be drawn at right angles to X′OX, intersecting the curve in H, and let points A′ and A be taken in X′OX, such that A′O = OA = a. Then OZ will be

* It will be seen that some of the proofs are only expressions, in geometrical form, of familiar methods of differentiation or integration.

called the *median* of the curve, X′OX the *base*, OH the *central ordinate*, and A′A the *parameter*.

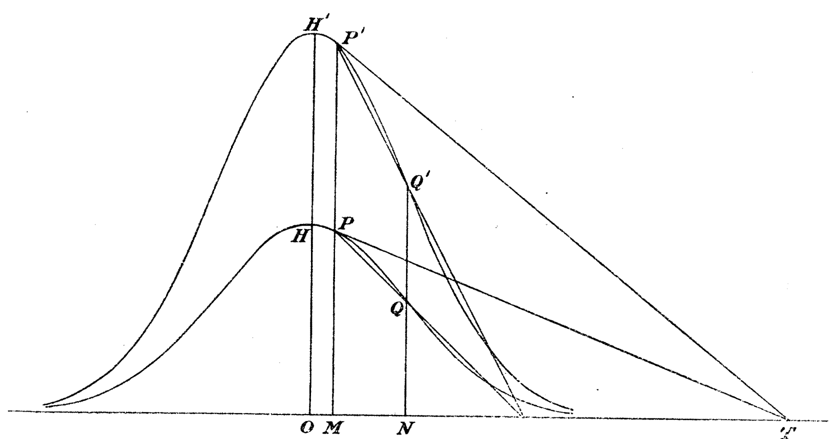
The curve is obviously symmetrical about the median, and asymptotic to the base in both directions.

The area bounded by the curve and the base will be called a *normal figure*.

§ 2. *Formation of Family of Curves by Projection.*—Let a new curve be formed by orthogonal projection of a normal curve with regard to the base in any ratio. Let MP and NQ be ordinates to the original curve, and MP′ and NQ′ the corresponding ordinates to the new curve (fig. 1). Then MP : MP′ :: NQ : NQ′. Hence PQ and P′Q′ will intersect on the base. Let N move up to and coincide with M. Then PQ and P′Q′ become the tangents at P and P′ to the two curves, and therefore these tangents meet the base in the same point T. Hence for the second curve we have also $OM.MT = OA^2$, and therefore this is also a normal curve of parameter A′A.

Similarly, if the curve is projected with regard to OZ in the ratio $a : b$, the new curve will be a normal curve of parameter $2b$, having the same median.

Fig. 1.



§ 3. *Limitation to Curves so obtained.*—Thus, by projection of a single normal curve with respect to the base and the median, we can get an indefinite number of normal curves of different parameters and different central ordinates. Conversely, if S and S′ are two normal curves placed so as to have the same base and the same median, either can be got from the other by projection. Let the parameters be $2a$ and $2b$ respectively. Project S into a curve S″ of parameter $2b$, and let Σ denote the family of projections of S″ with regard to the base. Then the tangent at each point of S′ coincides with the tangent to the particular curve of Σ which passes through this point. Hence S′ is one of the curves Σ, or else is the envelope of these curves. But the curves have no envelope at a finite distance. Hence S′ is a projection of S″.

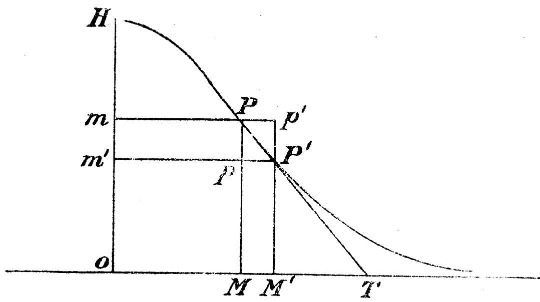
§ 4. *Standard Normal Curve.*—It is, therefore, convenient to take a standard normal curve, and to consider all other normal curves as obtained from it by projection.

For the standard form we take the curve whose semi-parameter is unity, and area unity. The central ordinate of this curve will for the present be denoted by C; we shall show later that $C = 1/\sqrt{2\pi}$. It is clear that if A is the area of a curve of parameter $2a$, its central ordinate is CA/a.

The curve may be traced by means of Table I. (p. 153). The second column of that table gives the ordinate of the standard curve in terms of the abscissa; the third gives its ratio to the central ordinate. Table II. (p. 155) is formed by inverting this latter table; it gives the abscissa in terms of the ratio of the ordinate to the central ordinate.

§ 5. *Moment-formulæ.*—Let MP, M′P′, be any two consecutive ordinates to a normal curve whose parameter is $2a$. Draw Pm and P′m' perpendicular to the central ordinate OH, and let p and p' be the intersections of MP, m'P′ and of M′P′, mP respectively (fig. 2). Then, if PP′ produced cuts the base in T, we have, by similar triangles,

Fig. 2.



$$Pp' . MP = P'p' . MT = pP . MT.$$

Hence

(1.) $OM \times \text{rectangle } MPp'M' = OM . Pp' . MP$
$= OM . MT \times pP = OM . MT\,(MP - M'P')$;

(2.) $OM^2 \times \text{rectangle } MPp'M' = OM . MT \times m'p . pP$
$= OM . MT \times \text{rectangle } m'pPm$;

(3.) $OM^{k+2} \times \text{rectangle } MPp'M' = OM . MT \times mP^k \times \text{rectangle } m'pPm.$

The kth moment of the rectangle $m'pPm$ about OH is $\frac{1}{k+1} . mP^k \times m'pPm$. Also when MM′ becomes indefinitely small, $OM . MT = a^2$. Hence, by summation, we see that

(i.) If MP and NQ are any two ordinates, the moment of the area MPQN about OH is $a^2\,(MP - NQ)$;

(ii$a$.) If Pm and Qn are the perpendiculars from P and Q on OH, the second moment of MPQN about OH is $a^2 \times$ area nQPm;

(ii*b*.) For the complete normal figure, the mean square of deviation from the mean is a^2;

(iii.) If λ_k denote the mean kth power of the deviation from the mean,

$$\lambda_{k+2} = (k+1)\,a^2\lambda_k = (k+1)\,\lambda_2\lambda_k,$$

which is the statistical equation to the curve.

This equation gives

$$\left.\begin{array}{l}\lambda_{2s-1} = 0 \\ \lambda_{2s} = (2s-1)(2s-3)\ldots 1\,.\,\lambda_2^s = \dfrac{\lfloor 2s}{2^s \lfloor s}\,\lambda_2^s\end{array}\right\}.$$

The Surface of Revolution of the Normal Curve.

§ 6. *Projective Solids and Surfaces.*—Let Σ be a surface whose equation referred to three rectangular axes OX, OY, OZ, is of the form $z = \phi(x)\,.\,\phi(y)$. Then if we take sections of Σ by a system of planes parallel to OZX, and project these sections on OZX, we obtain a system of curves which are the orthogonal projections of one another with regard to their common base OX. Similarly if we take sections by planes parallel to OZY. On this account it is convenient to call such a surface a *projective surface.* If the surface is terminated in all directions by the base-plane OXY, the volume included between this plane and the surface will be called a *projective solid.*

For the geometrical definition of a projective solid it is sufficient that the solid should be bounded by a plane base OXY, and that two lines OX, OY in this plane, at right angles to one another and to a line OZ, should be related to the solid in such a way that the sections of the surface by planes parallel to OZX, when projected on OZX, form a system of curves in orthogonal projection. If this is the case, it follows at once, from the elementary properties of projection, that the same property holds for sections by planes parallel to OZY.

The sections of the solid by the two sets of planes parallel to OZX and to OZY will be called *principal sections.*

The following properties of a projective solid are easily obtained from the geometrical definition.

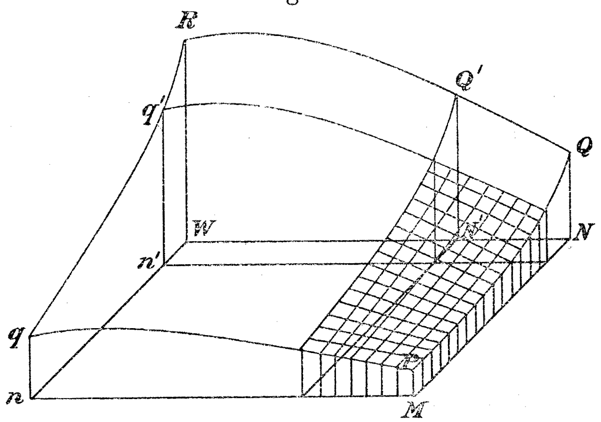
(i.) Let WR and MP be any two ordinates, and let the other ordinates in which the principal sections through WR and MP intersect be NQ and nq. Then WR.MP = NQ.nq.

(ii.) In one of the principal sections through an ordinate WR, take any two ordinates NQ and N′Q′; and in the other take any two ordinates nq and $n'q'$ (fig. 3). Draw the principal sections through these ordinates, and let them enclose (with the base and the upper surface) a volume V. Then WR.V = area NQQ′N′ × area $nqq'n'$.

(iii.) From (ii.) it follows that if we fix a principal section S, and take variable ordinates NQ and N′Q′, the volume of the solid bounded by the other principal sections through NQ and N′Q′ is proportional to the area NQQ′N′.

(iv.) From (ii.) it also follows that if V is the whole volume of the solid, WR any ordinate, and A and A′ the areas of the principal sections through WR, then WR.V = A.A′.

Fig. 3.



(v.) Let OH be the ordinate passing through the centre of gravity of the solid, and let S and S′ be the principal sections through OH. Then the central ordinates of all sections parallel to S (*i.e.*, the ordinates through their respective centres of gravity) lie in S′, and the central ordinates of all sections parallel to S′ lie in S.

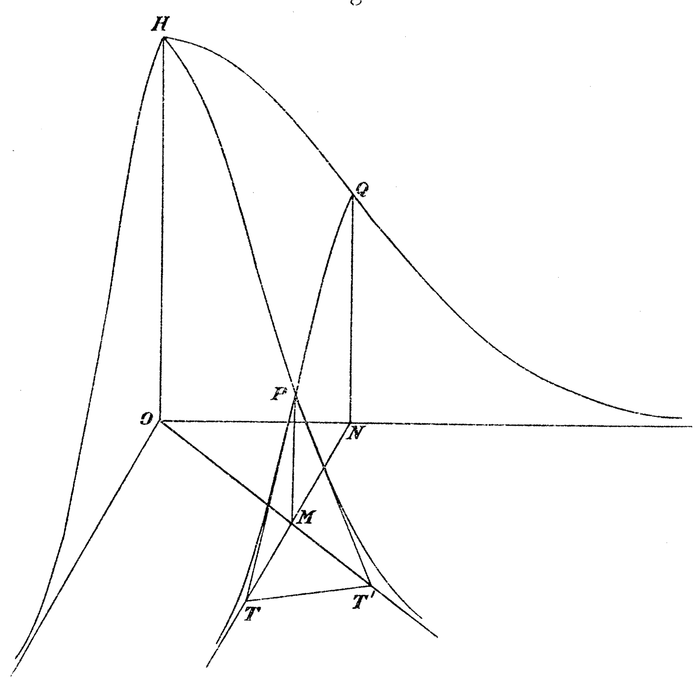
§ 7. *Normal Solid and Normal Surface.*—Let the half of a normal figure of parameter A′A = $2a$, lying on one side of the central ordinate OH, be rotated about this ordinate through four right angles. The solid so formed will be called a *normal solid*, and its surface will be called a *normal surface*. The plane traced out by the base will be called the *base-plane*. A section of the solid by a plane perpendicular to the base-plane will be called a *vertical* section.

§ 8. *Normal Solid is Projective Solid.*—Let S be any vertical section of the solid, and MP any ordinate in this section. Draw ON perpendicular to the plane of the section, and let NQ be the ordinate at N. Let the tangents at P to the section S, and to the central section through MP (*i.e.*, the section through MP and the axis), cut the base-plane in T and T′ respectively (fig. 4).

Since PT and PT′ are tangents to sections through P, the plane PTT′ is the tangent plane to the solid at P. But the solid is a solid of revolution, and therefore this plane is perpendicular to the plane OMP. The base-plane is also perpendicular to the plane OMP, and therefore the intersection TT′ is perpendicular to this latter plane. Hence OT′T is a right angle, and therefore a circle goes round ONT′T, so that NM.MT = OM.MT′.

But the section by the plane OMP is a normal figure of parameter $2a$, and therefore OM.MT′ = a^2. Hence also NM.MT = a^2; *i.e.*, the section S is a normal figure of parameter $2a$, having NQ for its central ordinate.

Fig. 4.



Thus every vertical section of the solid is a normal figure of the same parameter, having its central ordinate in the plane through the axis at right angles to the plane of the section.

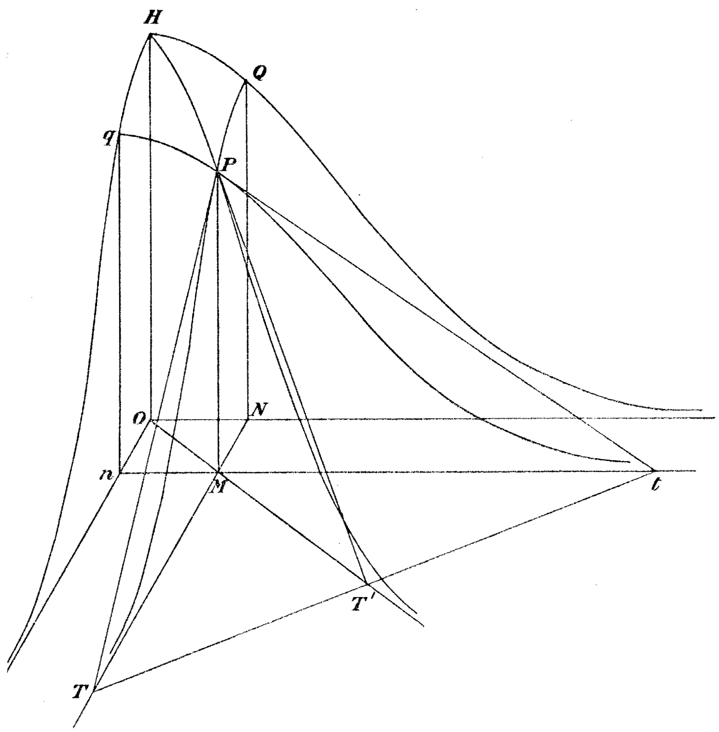
It follows from § 3 that the solid is a projective solid, any two vertical sections at right angles to one another being regarded as principal sections.

§ 9. *Converse Propositions.*—There are two converse propositions.

(i.) If two principal sections of a projective solid are normal figures of equal parameter, the solid is one of revolution.

Let this parameter be $2a$. From § 2 it follows that every principal section is a normal figure of parameter $2a$. The solid will obviously have a maximum ordinate OH; and each of the two principal sections through OH will contain the central ordinates of all sections by planes perpendicular to it. Take any other section through OH; and let MP be any ordinate in this section. Draw planes through MP cutting the principal sections through OH in ordinates NQ and nq. Then the sections NQPM and nqPM are normal figures of parameter $2a$, having NQ and nq for their central ordinates. Let the tangents to these sections and to the section OHPM cut the respective bases in T, t, T′ (fig. 5). Then PT, PT′, Pt all lie in the tangent plane to the surface at P, and therefore TT′t is a straight line. Also $\mathrm{NM}.\mathrm{MT} = a^2 = n\mathrm{M}.\mathrm{M}t$, so that ON : NM :: TM : Mt. Hence the triangles ONM, TMt are similar, and angle MTt = angle NOM; and therefore a circle goes round NOTT′. Hence $\mathrm{OM}.\mathrm{MT}' = \mathrm{NM}.\mathrm{MT} = a^2$, and therefore the section OHPM is a normal figure of parameter $2a$, having OH for its central ordinate. This is true for every section through OH, and therefore the solid is one of revolution.

Fig. 5.



(ii.) If a solid of revolution is also a projective solid, the generating figure is a normal figure.

Let OH be the central ordinate. Then every vertical section is symmetrical about the plane through OH perpendicular to it, and any two vertical sections, if arranged so as to have their central ordinates coincident, will be interconvertible by projection. Let S be any section through OH, and let NQ and N′Q′ be any two ordinates in this section, ON being greater than ON′. Let the tangents to S at Q and Q′ cut ON′ N in T and T′.

Describe a circle in the base-plane on ON as diameter, and draw the chord NM = ON′. Draw the ordinate MP, and let the tangent at Q to the section MPQN cut MN produced in R (fig. 6). Then MP is the central ordinate of the section MPQN; and therefore, since this section and the section OHQ′N′ are interconvertible by projection, it follows that NR = N′T′.

Since QR and QT are tangents to sections through NQ, QRT is the tangent plane at Q. The solid being a solid of revolution about OH, this tangent plane must be perpendicular to the plane OQT. The base-plane is also perpendicular to the plane OQT, and therefore TR, which is the line of intersection of the tangent plane and the base-plane, is perpendicular to the plane OQT. Hence OTR is a right angle, and therefore a circle goes round OMTR, so that ON . NT = MN . NR = ON′ . N′T′. In other words, the rectangle ON . NT is constant for different positions of N, and therefore the central section is a normal figure.

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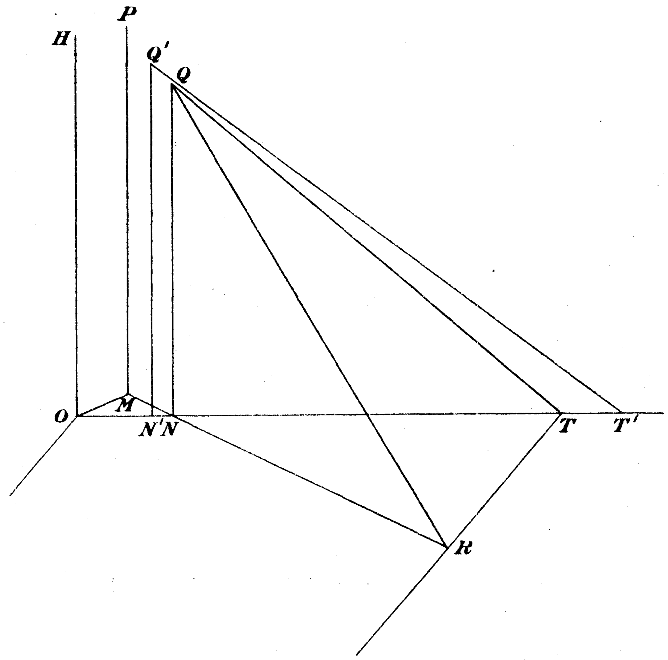
Fig. 6.



§ 10. *Value of* C.—Let A and A′ be the areas of two sections through OH at right angles to one another; and let V be the whole volume of the solid. Then, since the solid is a projective solid, $\mathrm{OH}\,.\,\mathrm{V} = \mathrm{A}\,.\,\mathrm{A}' = \mathrm{A}^2$ (§ 6 (iv.)); and, since it is a solid of revolution, $\mathrm{V} = 2\pi a^2\,.\,\mathrm{OH}$ (§ 5 (i.), and GULDINUS' theorem). But $\mathrm{OH} = \mathrm{CA}/a$ (§ 4). Hence $\mathrm{C} = 1/\sqrt{2\pi}$.

It is convenient to consider the solid as obtained from a standard form by an orthogonal and an axial* projection. As the standard solid we shall take the solid whose volume is unity and whose vertical sections are normal figures of semi-parameter unity. The central ordinate of this solid is $1/2\pi$.

§ 11. *Representation of Segment of Normal Solid by an Area.*—Let Σ be any closed curve in the base of a normal solid, whose principal ordinate is OH, and whose parameter is $2a$; and let V be the portion of the solid which lies above Σ, *i.e.*, which is bounded by Σ, by the surface of the solid, and by a cylinder K of which Σ is a normal section. We require a method of determining the volume V.

Let Σ′ be the upper boundary of V, *i.e.*, the area cut out of the surface of the normal solid by the cylinder K. Describe a circular cylinder of radius b, and of height OH, having OH as axis; and project Σ′ on this cylinder by lines perpendicular to OH. The projection will be a closed curve σ. Now the volume V can be divided into elements by a series of planes through OH at indefinitely small angular distances from one another. Let Π and Π′ be two consecutive planes of the system,

* By an axial projection of a surface or a solid with regard to a straight line is meant the surface or solid obtained by projecting every point orthogonally with regard to this straight line in a definite ratio.

the angle between them being θ; let them cut σ in the straight lines pq and $p'q'$, and let Π cut V in the area MPQN, bounded by the ordinates MP and NQ. Then $pq = \mathrm{NQ} \backsim \mathrm{MP}$; and therefore, by § 5, the moment of the area MPQN about OH is equal to $a^2 . pq$. Hence, by GULDINUS' theorem, the portion of V included between Π and Π' is equal to $a^2 . pq . \theta = a^2/b \times$ area $pq\,q'p'$. By summation, we see that $\mathrm{V} = a^2/b \times$ area σ.

The cylinder, with the curve σ, may be supposed to be unwrapped on a plane. Hence when we are given the central section of the solid, and a plan showing the form of Σ and its position with regard to O, we are able to construct, by geometrical methods, a curve whose area will give us the volume V. Take a standard line OX on the plan. Through O draw a line inclined to OX at an angle whose circular measure is α, and let this line cut Σ in points M and N. Take abscissæ OM and ON along the base of the given central section, and draw the ordinates MP and NQ. On a line O′X′ take $\mathrm{O'L'} = b\alpha$, and draw an ordinate $\mathrm{L'}qp$ such that $\mathrm{L'}p = \mathrm{MP}$, $\mathrm{L'}q = \mathrm{NQ}$. The different points p and q corresponding to different values of α will form a curve, whose area can be measured; and this area, multiplied by a^2/b, is the volume required.*

If the curve Σ encloses the base of the principal ordinate OH, the continuity of the boundary of σ will be broken when the cylinder is unwrapped. The locus of the points p is then the top of the rectangle representing the complete cylinder, and the area to be taken is the area between this, the sides of the rectangle, and the curve which is the locus of q. Similarly, if any portion of the boundary of Σ is at infinity, the corresponding part of the boundary of σ will lie along the base of the rectangle representing the complete cylinder.

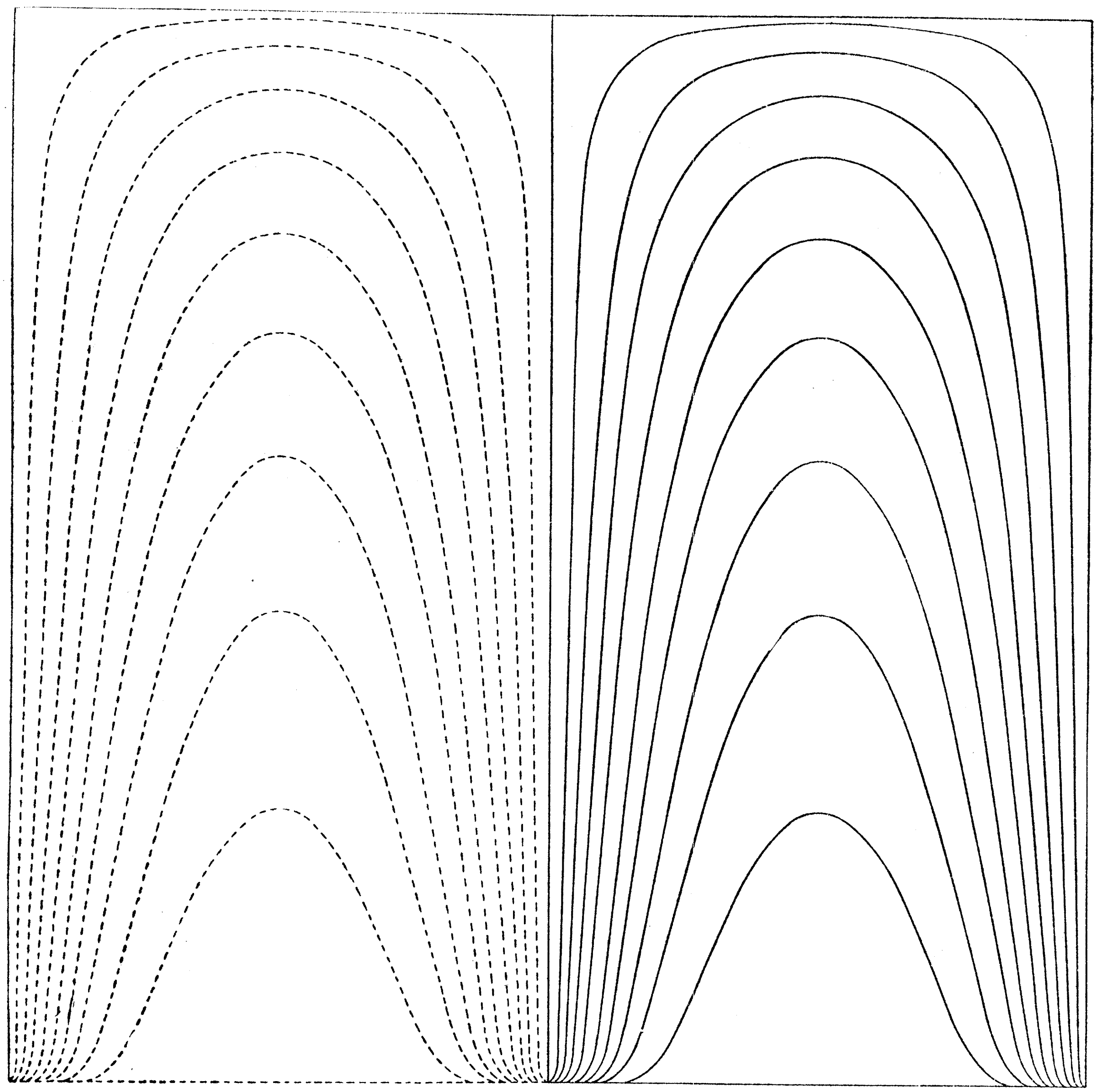
The area σ is unaltered by projecting it at right angles to O′X′ in the ratio $1 : \lambda$, and parallel to O′X′ in the ratio $\lambda : 1$. Thus we shall have $\mathrm{L'}p = \lambda . \mathrm{MP}$, $\mathrm{L'}q = \lambda . \mathrm{NQ}$, the point L′ being taken so that $\mathrm{O'L'} = b\alpha/\lambda$. When the solid is the standard solid, it is convenient to take $b = a$ $(= 1)$, and $\lambda = 2\pi$; the unwrapped cylinder then becomes a square whose base is unity and height unity; and the values of $\mathrm{L'}p$ and $\mathrm{L'}q$ are given by the third column of Table I. (p. 153).

If, for example, we divide the standard solid into twenty equal portions by nineteen parallel vertical planes, and if the cylinder is supposed to be divided along one of the lines in which it is cut by the central plane, and then unwrapped, and projected vertically in the ratio of $1 : 2\pi$ and horizontally in the ratio of $2\pi : 1$, we

* Generally, let V be a portion cut out of a solid of revolution by a closed cylinder K, whose generating lines are parallel to the axis of revolution. Let F denote the section of the solid by a plane through the axis of revolution; and let S be a curve lying in the plane of F and related to it in such a way that any ordinate MP (drawn to S from a base at right angles to the axis of revolution) is proportional to the moment, about the axis, of that portion of F which lies beyond MP. Then, if F is given geometrically, and if the section of the cylinder K and its position with regard to the axis are given, we can construct a figure whose area will be proportional to the volume V.

shall obtain the figure shown in fig. 7. The figure consists of two similar portions, each of which is divided into ten equal parts by nine curves; each curve touching the corresponding half of the base at its extremities, and being symmetrical about

Fig. 7.



its central ordinate. The curves may be traced by means of Tables III. and IV. (pp. 156–158); Table III. gives the ordinates in terms of the abscissa, measured from the extremity of the base of the figure; and Table IV. is a converse table, giving the abscissæ of the different curves in terms of the ordinate.*

* The values in Table IV. were calculated by means of CALLET's tables, in which the quadrant is divided centesimally.

General Theorems Relating to Normal Distributions.

§ 12. *Mean Squares and Mean Products of Composite Measures.*—Let A, B, C, . . . E, F, G be a number of attributes, all of which exist in every member of a community; and let the measures of their respective magnitudes be denoted by L, M, N, . . . P, Q, R. Let the mean values of L, M, N, . . . P, Q, R be respectively L_1, M_1, N_1, . . . P_1, Q_1, R_1; let the mean squares of their deviations from their respective means be a^2, b^2, c^2, . . . e^2, f^2, g^2; and let the mean product of the deviations of any two L and M from their respective means be denoted by S (L, M). Then, whatever the relations amongst the distributions may be,

(i.) The mean value of $lL + mM + nN \ldots + rR$, where $l, m, n, \ldots r$ are any constants, is $lL_1 + mM_1 + nM_1 + \ldots + rR_1$; and the mean square of its deviation from its mean is

$$l^2a^2 + m^2b^2 + n^2c^2 + \ldots + r^2g^2 + 2lm\mathrm{S}(\mathrm{L}, \mathrm{M}) + 2ln\mathrm{S}(\mathrm{L}, \mathrm{N}) + 2mn\mathrm{S}(\mathrm{M}, \mathrm{N}) + \ldots$$

(ii.) The mean product of the deviations of $lL + mM + nN + \ldots + rR$ and $l'L + m'M + n'N + \ldots + r'R$ from their respective means is

$$ll'a^2 + mm'b^2 + nn'c^2 + \ldots + rr'g^2 + (lm' + l'm)\,\mathrm{S}(\mathrm{L}, \mathrm{M}) + (ln' + l'n)\,\mathrm{S}(\mathrm{L}, \mathrm{N}) + (mn' + m'n)\,\mathrm{S}(\mathrm{M}, \mathrm{N}) + \ldots$$

As we shall often require to use these last two expressions, it will be found convenient to express the mean squares and mean products in the form of a table, thus:—

	L	M	N	&c.
L	a^2	S (L, M)	S (L, N)	
M	S (L, M)	b^2	S (M, N)	
N	S (L, N)	S (M, N)	c^2	
&c.				

§ 13. *Independent Normal Distributions.*—If the different values of L, in the class distinguished by particular values of M, N, . . . P, Q, R, are distributed in the same way, whatever these particular values may be, the distribution of L is said to be independent of the distributions of M, N, . . . P, Q, R.

If the distribution of Q is independent of that of R; the distribution of P independent of those of Q and R; and so on, for L, M, N, . . . P, Q, R: then the distributions of L, M, N, . . . P, Q, R may be said to be mutually independent.

Now suppose that each distribution, taken separately, is normal; we require to find the distribution of $l\text{L} + m\text{M} + n\text{N} + \ldots + p\text{P} + q\text{Q} + r\text{R}$, where $l, m, n, \ldots p, q, r$ are any constants.

Consider first the case of two measures L and M. Let their mean values be L_1 and M_1, and let their mean squares of deviation from the mean be a^2 and b^2. Let $\text{L} = \text{L}_1 + ax$, $\text{M} = \text{M}_1 + by$. Then the values of x and of y are distributed normally about mean values zero with mean squares unity, and the distribution of x is independent of the distribution of y. Take two lines OX, OY at right angles to one another, and on OXY as base-plane construct the solid of frequency of values of x and y, these values being measured parallel to OX and OY respectively. Let OZ be drawn at right angles to OXY; and let K_1 and K_2 be two planes whose equations referred to OX, OY, OZ as axes are $la\,.\,x + mb\,.\,y = \xi_1$ and $la\,.\,x + mb\,.\,y = \xi_2$ respectively, where ξ_1 and ξ_2 have any values. Then the portion of the solid lying between K_1 and K_2 includes all elements representing individuals for which $la\,.\,x + mb\,.\,y$ lies between ξ_1 and ξ_2; and therefore the number of these individuals is proportional to the volume of this portion of the solid. Denote this volume by V.

Since the distribution of x is independent of the distribution of y, the sections of the solid of frequency by planes parallel to OZX are figures which when projected on OZX are orthogonal projections of one another with regard to OX; in other words, the solid is a projective solid. Since the values of x are distributed normally with mean value zero and mean square unity, it follows from (iii.) of § 6 that the sections by planes parallel to OZX are normal figures whose semi-parameters are unity, and whose central ordinates lie in OZY; and similarly the sections by planes parallel to OZY are normal figures whose semi-parameters are unity and whose central ordinates lie in OZY. Hence, by § 9 (i.), the solid is a normal solid; and therefore it may be regarded as a projective solid whose principal sections are parallel and perpendicular to the planes K_1 and K_2. Through OZ draw a plane at right angles to K_1 and K_2, cutting them in ordinates W_1R_1 and W_2R_2, and cutting the solid in a normal figure S. Then the volume V is proportional to the area $\text{W}_1\text{R}_1\text{R}_2\text{W}_2$ of the figure S. Also $\text{OW}_1 = \xi_1/\sqrt{l^2a^2 + m^2b^2}$, $\text{OW}_2 = \xi_2/\sqrt{l^2a^2 + m^2b^2}$. Hence the number of individuals for which $la.x + mb.y$ lies between ξ_1 and ξ_2 is proportional to the area, comprised between ordinates at distances $\xi_1/\sqrt{l^2a^2 + m^2b^2}$ and $\xi_2/\sqrt{l^2a^2 + m^2b^2}$ from the median, of a normal figure of semi-parameter unity; and therefore, by § 2, it is proportional to the area, comprised between ordinates at distances ξ_1 and ξ_2 from the median, of a normal figure of semi-parameter $\sqrt{l^2a^2 + m^2b^2}$. In other words, the values of $la.x + mb.y$ are distributed normally with mean square $l^2a^2 + m^2b^2$ about a mean value zero, and therefore the values of $lL + mM$ are distributed normally with this mean square* about a mean value $lL_1 + mM_1$.

Next take the more general case. Since the distributions of Q and of R are independent and normal, the distribution of $qQ + rR$ is normal. Again, since the distribution of P is independent of the distributions of Q and R, it is independent of the distribution of $qQ + rR$; and therefore, since the distribution of P is normal, the distribution of $pP + qQ + rR$ is normal. Proceeding in this way, we see that if the distributions of L, M, N, ... P, Q, R are mutually independent, and if each distribution, taken separately, is normal, the distribution of $lL + mM + nN + \ldots + pP + qQ + rR$ is also normal.

We might have obtained this result from the *statistical equation* of the normal curve (§ 5). Let $L - L_1 = L'$, $M - M_1 = M'$, $N - N_1 = N'$, ... Also let $S(L'^{\alpha}M'^{\beta}N'^{\gamma}\ldots)$ denote the mean value of $L'^{\alpha}M'^{\beta}N'^{\gamma}\ldots$, and let λ_k denote the mean value of $(lL' + mM' + nN' + \ldots)^k$. Then, since the distributions are independent, $S(L'^{\alpha}M'^{\beta}N'^{\gamma}\ldots) = S(L'^{\alpha}).S(M'^{\beta}).S(N'^{\gamma})\ldots$. Also, by § 5, $S(L'^{2s-1}) = 0$, and $S(L'^{2s}) = \frac{\lfloor 2s}{2^s \lfloor s} a^{2s}$; and similarly for M', N', ... Hence we see that—

(i.) Every term in the expansion of $(lL' + mM' + nN' + \ldots)^{2s-1}$ must contain an odd power of one at least of the quantities L', M', N', ...; and therefore, by taking the mean, $\lambda_{2s-1} = 0$;

(ii.) $\lambda_2 = l^2a^2 + m^2b^2 + n^2c^2 + \ldots$

(iii.) $\lambda_{2s} =$ mean value of $(lL' + mM' + nN' + \ldots)^{2s}$

$$= \Sigma\Sigma\Sigma\ldots \frac{\lfloor 2s}{\lfloor 2\alpha \lfloor 2\beta \lfloor 2\gamma \ldots} S\{(lL')^{2\alpha}\}.S\{(mM')^{2\beta}\}.S\{(nN')^{2\gamma}\}\ldots$$

(the summation being made for all positive integral values of α, β, γ, ... satisfying the condition $\alpha + \beta + \gamma + \ldots = s$)

$$= \Sigma\Sigma\Sigma\ldots \frac{\lfloor 2s}{\lfloor 2\alpha \lfloor 2\beta \lfloor 2\gamma \ldots} l^{2\alpha}m^{2\beta}n^{2\gamma}\ldots \frac{\lfloor 2\alpha}{2^{\alpha}\lfloor \alpha} a^{2\alpha} \cdot \frac{\lfloor 2\beta}{2^{\beta}\lfloor \beta} b^{2\beta} \cdot \frac{\lfloor 2\gamma}{2^{\gamma}\lfloor \gamma} c^{2\gamma}\ldots$$

$$= \frac{\lfloor 2s}{2^s\lfloor s} \Sigma\Sigma\Sigma\ldots \frac{\lfloor s}{\lfloor \alpha \lfloor \beta \lfloor \gamma \ldots} (l^2a^2)^{\alpha}.(m^2b^2)^{\beta}.(n^2c^2)^{\gamma}\ldots$$

$$= \frac{\lfloor 2s}{2^s\lfloor s}(l^2a^2 + m^2b^2 + n^2c^2 + \ldots)^s = \frac{\lfloor 2s}{2^s\lfloor s}\lambda_2^s;$$

and therefore, for all positive integral values of k,

$$\lambda_{k+2} = (k+1)\lambda_2\lambda_k.$$

* The expression "mean square" may generally be used, without confusion, to denote the mean square of deviation from the mean.

Hence the values of $lL' + mM' + nN' + \ldots$ are normally distributed; and therefore the values of $lL + mM + nN + \ldots$ are normally distributed.

§ 14. *Correlated Normal Distributions.*—If L, M, N, ... R are the measures of coexistent attributes A, B, C, ... G; and if the values of L, in every class distinguished by particular values of M, N, ... R, are distributed normally with constant mean square about a mean value $L_1 + \mu(M - M_1) + \nu(N - N_1) + \ldots + \rho(R - R_1)$, where L_1, M_1, N_1, ... R_1 are the respective mean values of L, M, N, ... R taken separately, and μ, ν, ... ρ are constants: then the distribution of L is said to be correlated with the distributions of M, N, ... R.

If the distribution of R is normal; the distribution of Q correlated with that of R; the distribution of P correlated with those of Q and R; and so on, for L, M, N, ... P, Q, R: then the distributions of L, M, N, ... P, Q, R may be said to be mutually correlated. We require to find, in this case, the distribution of $lL + mM + nN + \ldots + pP + qQ + rR$, where $l, m, n, \ldots p, q, r$ are any constants.

For convenience, consider only the case of four attributes L, M, N, R. From the definition, we see that $L - L_1$ is equal to $\mu(M - M_1) + \nu(N - N_1) + \rho(R - R_1) + L'$, where L' is independent of $M - M_1$, $N - N_1$, and $R - R_1$, and is distributed normally with mean value zero. Similarly $M - M_1$ is equal to $\nu'(N - N_1) + \rho'(R - R_1) + M'$, where M' is independent of $N - N_1$ and $R - R_1$; and $N - N_1$ is equal to $\rho''(R - R_1) + N'$, where N' is independent of $R - R_1$; the values of M' and of N' being distributed normally with mean values zero. Since M' is independent of $N - N_1$ and $R - R_1$, and $N - N_1$ is equal to $\rho''(R - R_1) + N'$, it follows that M' is independent of N' and $R - R_1$; and similarly L' is independent of M', N', and $R - R_1$. Thus the distributions of L', M', N', and $R - R_1$ are mutually independent. Also each of the measures $L - L_1$, $M - M_1$, $N - N_1$, $R - R_1$, is a linear function of the measures L', M', N', $R - R_1$; and therefore $l(L - L_1) + m(M - M_1) + n(N - N_1) + r(R - R_1)$ is a linear function of these measures It follows, from § 13, that the values of $l(L - L_1) + m(M - M_1) + n(N - N_1) + r(R - R_1)$ are normally distributed; *i.e.*, the values of $lL + mM + nN + rR$ are normally distributed. The argument obviously applies to any number of correlated distributions.

This result might also be obtained by the second of the two methods given in the last section.

II. Theory of Error.

§ 15. *Distribution of linear function of errors of random selection.*—Let the individuals comprised in an indefinitely great community be divided into any number of classes A, B, C, ..., and let the numbers in these classes be proportional to $\alpha, \beta, \gamma, \ldots$, so that $\alpha + \beta + \gamma + \ldots = 1$. Suppose a random selection of n individuals to be made, and let the numbers drawn from the different classes be respectively $n\alpha', n\beta', n\gamma', \ldots$, so that $\alpha' + \beta' + \gamma' + \ldots = 1$. Then $\alpha' - \alpha$, $\beta' - \beta$, $\gamma' - \gamma, \ldots$ are the *errors* in $\alpha, \beta, \gamma, \ldots$ We require to investigate the distribution of the different values of $a(\alpha' - \alpha) + b(\beta' - \beta) + c(\gamma' - \gamma) + \ldots$ for different random selections of n individuals, $a, b, c, \ldots$ being any constants.

(1.) If we only require the mean and the mean square, we can most conveniently use the formulæ of § 12. Suppose an indefinitely great number of random selections to be made. Then the proportion of cases in which p come from A and the remaining $n - p$ from the other classes is

$$\frac{\lfloor n}{\lfloor p \lfloor n-p} \alpha^p (1-\alpha)^{n-p}.$$

Hence

(i.) the mean value of α' is

$$\sum_{p=0}^{p=n} \frac{\lfloor n}{\lfloor p \lfloor n-p} \alpha^p (1-\alpha)^{n-p} \cdot \frac{p}{n} = \alpha \sum_{p=1}^{p=n} \frac{\lfloor n-1}{\lfloor p-1 \lfloor n-p} \alpha^{p-1} (1-\alpha)^{n-p} = \alpha;$$

so that the mean value of $\alpha' - \alpha$ is zero; and

(ii.) the mean square of α' is

$$\sum_{p=0}^{p=n} \frac{\lfloor n}{\lfloor p \lfloor n-p} \alpha^p (1-\alpha)^{n-p} \cdot \frac{p^2}{n^2} = n^{-2} \sum_{p=0}^{p=n} \frac{\lfloor n}{\lfloor p \lfloor n-p} \alpha^p (1-\alpha)^{n-p} \{p(p-1) + p\}$$

$$= n^{-2}\{n(n-1)\alpha^2 + n\alpha\} = \alpha^2 + \alpha(1-\alpha)/n;$$

so that the mean square of $\alpha' - \alpha$ is $\alpha(1-\alpha)/n$.

(iii.) Similarly the mean value of $\alpha' \beta'$ is

$$\sum_{p=0}^{p=n} \sum_{q=0}^{q=n} \frac{\lfloor n}{\lfloor p \lfloor q \lfloor n-p-q} \alpha^p \beta^q (1-\alpha-\beta)^{n-p-q} \cdot \frac{p}{n} \cdot \frac{q}{n}$$

$$= \frac{n(n-1)}{n^2} \alpha\beta \sum_{p=1}^{p=n} \sum_{q=1}^{q=n} \frac{\lfloor n-2}{\lfloor p-1 \lfloor q-1 \lfloor n-p-q} \alpha^{p-1} \beta^{q-1} (1-\alpha-\beta)^{n-p-q}$$

$$= \alpha\beta - \alpha\beta/n;$$

and therefore the mean product of $\alpha' - \alpha$ and $\beta' - \beta$ is $-\alpha\beta/n$. From these three results it follows that

(iv.) the mean value of $a(\alpha' - \alpha) + b(\beta' - \beta) + c(\gamma' - \gamma) + \ldots$ is zero;

(v.) the mean square is

$$a^2\alpha(1-\alpha)/n + b^2\beta(1-\beta)/n + c^2\gamma(1-\gamma)/n + \ldots$$
$$- 2ab\alpha\beta/n - 2ac\alpha\gamma/n - 2bc\beta\gamma/n - \ldots$$
$$= \{(a^2\alpha + b^2\beta + c^2\gamma + \ldots) - (a\alpha + b\beta + c\gamma + \ldots)^2\}/n;$$

(vi.) the mean product of $a(\alpha' - \alpha) + b(\beta' - \beta) + c(\gamma' - \gamma) + \ldots$ and $a'(\alpha' - \alpha) + b'(\beta' - \beta) + c'(\gamma' - \gamma) + \ldots$ is

$$\{(aa'\alpha + bb'\beta + cc'\gamma + \ldots) - (a\alpha + b\beta + c\gamma + \ldots)(a'\alpha + b'\beta + c'\gamma + \ldots)\}/n.$$

(2.) Let λ_k denote the mean kth power of $a(\alpha' - \alpha) + b(\beta' - \beta) + c(\gamma' - \gamma) + \ldots$ The proportion of cases in which the numbers drawn from the different classes are $p, q, r \ldots$, where $p + q + r + \ldots = n$, is

$$\frac{\lfloor p + q + r + \ldots}{\lfloor p \lfloor q \lfloor r \ldots} \alpha^p \beta^q \gamma^r \ldots$$

Hence the mean kth power of $a\alpha' + b\beta' + c\gamma' + \ldots$ is

$$n^{-k}\Sigma\Sigma\Sigma \ldots \frac{\lfloor p + q + r + \ldots}{\lfloor p \lfloor q \lfloor r \ldots} \alpha^p\beta^q\gamma^r \ldots (ap + bq + cr + \ldots)^k$$

$$= n^{-k} \lfloor k \times \text{coefficient of } \theta^k \text{ in } \Sigma\Sigma\Sigma \ldots \frac{\lfloor p + q + r + \ldots}{\lfloor p \lfloor q \lfloor r \ldots} \alpha^p\beta^q\gamma^r \ldots e^{(ap+bq+cr+\ldots)\theta}$$

$$= n^{-k} \lfloor k \times \text{co. } \theta^k \text{ in } \Sigma\Sigma\Sigma \ldots \frac{\lfloor p + q + r + \ldots}{\lfloor p \lfloor q \lfloor r \ldots} (\alpha e^{a\theta})^p . (\beta e^{b\theta})^q . (\gamma e^{c\theta})^r \ldots$$

$$= \lfloor k \times \text{co. } \theta^k \text{ in } (\alpha e^{a\theta/n} + \beta e^{b\theta/n} + \gamma e^{c\theta/n} + \ldots)^n.$$

Denote $a\alpha + b\beta + c\gamma + \ldots$ by ω. Then, since $\alpha' + \beta' + \gamma' + \ldots = 1$,

$$a(\alpha' - \alpha) + b(\beta' - \beta) + c(\gamma' - \gamma) + \ldots$$
$$= a\alpha' + b\beta' + c\gamma' + \ldots - \omega(\alpha' + \beta' + \gamma' + \ldots)$$
$$= (a - \omega)\alpha' + (b - \omega)\beta' + (c - \omega)\gamma' + \ldots$$

Hence, writing $a - \omega$, $b - \omega$, $c - \omega$, ... for $a, b, c, \ldots$, in the above result, we see that

$$\lambda_k = \lfloor k \times \text{coefficient of } \theta^k \text{ in } \{\alpha e^{(a-\omega)\theta/n} + \beta e^{(b-\omega)\theta/n} + \gamma e^{(c-\omega)\theta/n} + \ldots\}^n.$$

§ 16. *Tendency of Distribution to become Normal.*—We have now to prove that, when n becomes very great, the distribution of values of $a(\alpha' - \alpha) + b(\beta' - \beta) + c(\gamma' - \gamma) + \ldots$ tends to become normal. To do this, we can use either the geometrical or the statistical definition of the normal curve. Of the two methods, the latter is the simpler.

(1.) Since the mean square of $a(\alpha' - \alpha) + b(\beta' - \beta) + c(\gamma' - \gamma) + \ldots$ varies inversely as n, it is more convenient to find the distribution of

$$\sqrt{n}\,\{a(\alpha' - \alpha) + b(\beta' - \beta) + c(\gamma' - \gamma) + \ldots\}.$$

Let the mean kth power of this last expression be denoted by μ_k, so that

$$\mu_2 = (a^2\alpha + b^2\beta + c^2\gamma + \ldots) - (a\alpha + b\beta + c\gamma + \ldots)^2.$$

By expanding the expression at the end of § 15, and writing $n\theta$ for θ, we see that

$$\mu_k = n^{-\frac{1}{2}k} \lfloor k \times \text{coefficient of } \theta^k \text{ in } \{1 + \tfrac{1}{2}\mu_2\theta^2 + C_3\theta^3 + C_4\theta^4 + \ldots\}^n,$$

where $C_3, C_4, \ldots$ are functions of $a, b, c, \ldots, \alpha, \beta, \gamma, \ldots$ Denote $\frac{1}{2}\mu_2\theta^2 + C_3\theta^3 + C_4\theta^4 + \ldots$ by Θ, and expand $(1 + \Theta)^n$ by the binomial theorem. Then the highest power of n contained in μ_k comes from the term involving $\Theta^{\frac{1}{2}k}$ when k is even, or from the term involving $\Theta^{\frac{1}{2}(k-1)}$ when k is odd. Hence, when n is made indefinitely great,

$$\left.\begin{aligned} \mu_{2s} &= n^{-s} \lfloor 2s \times \frac{n^s}{\lfloor s} (\tfrac{1}{2}\mu_2)^s = \frac{\lfloor 2s}{2^s \lfloor s} \mu_2^{\,s} \\ \mu_{2s+1} &= n^{-s-\frac{1}{2}} \lfloor 2s+1 \times \frac{n^s}{\lfloor s} \cdot s\,(\tfrac{1}{2}\mu_2)^{s-1}\, C_3 = 0 \end{aligned}\right\},$$

and therefore the distribution is ultimately normal.

It follows that the distribution of values of $a(\alpha' - \alpha) + b(\beta' - \beta) + c(\gamma' - \gamma) + \ldots$ is also normal.

It will be noticed that, when n is finite, the number of terms in μ_{2s} or μ_{2s+1} increases with s, and becomes infinite when s is infinite. Thus the approximation of the actual distribution to the ultimate normal distribution is close as regards the low moments, n being supposed to be moderately great, but is not close as regards very high moments. The difference between the two distributions is therefore due mainly to the values of $\sqrt{n}\,\{a(\alpha' - \alpha) + b(\beta' - \beta) + c(\gamma' - \gamma) + \ldots\}$ which are great in comparison with $\sqrt{\mu_2}$. But these are values which only occur very rarely; and therefore, for practical purposes, we may regard the two distributions as identical.

(2.) To obtain the same result from the geometrical definition of the curve, we must use § 14.

(i.) To find the distribution of values of $\sqrt{n}\,(\alpha' - \alpha)$, we take a series of points $M_0, M_1, \ldots M_n$, at equal distances $1/\sqrt{n}$ along a straight line X′X; and then draw ordinates M_0P_0, M_1P_1, ... M_nP_n equal to the coefficients in the expansion of $\sqrt{n}\,(\beta x + \alpha y)^n$, where $\alpha + \beta = 1$. Thus

$$M_pP_p = \sqrt{n}\,.\,\alpha^p\beta^{n-p}C_p^n,$$

where C_p^n stands for $\dfrac{\lfloor n}{\lfloor p \lfloor n-p}$. Then, if n is increased indefinitely, the locus of the points $P_0, P_1, \ldots P_n$ will be a curve, which will be the curve of frequency of values of $\sqrt{n}\,(\alpha' - \alpha)$.

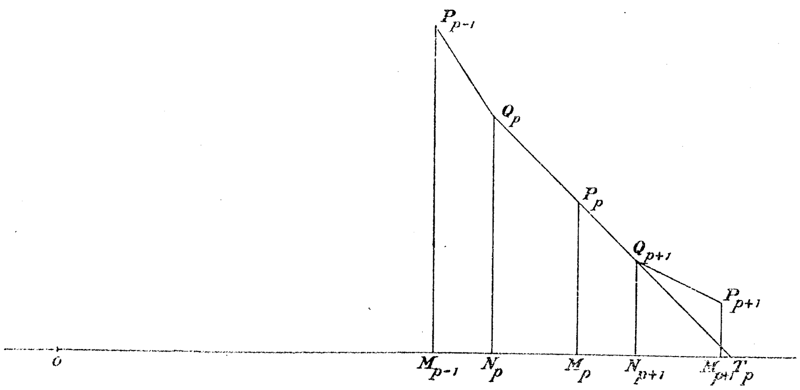
To find this curve, take a second series of points N_0, N_1, ... N_{n+1}, also at equal distances $1/\sqrt{n}$, and in such a position with regard to the former series that

$$M_{p-1}N_p = \alpha/\sqrt{n}, \qquad N_pM_p = \beta/\sqrt{n};$$

and at the points N_1, N_2, ... N_n erect ordinates N_1Q_1, N_2Q_2, ... N_nQ_n (fig. 8) equal to the coefficients in the expansion of $\sqrt{n}\,(\beta x + \alpha y)^{n-1}$. Thus

$$\left.\begin{array}{l} N_pQ_p = \sqrt{n}\,.\,\alpha^{p-1}\beta^{n-p}C^{n-1}_{p-1} \\ N_{p+1}Q_{p+1} = \sqrt{n}\,.\,\alpha^{p}\beta^{n-p-1}C^{n-1}_{p} \end{array}\right\}.$$

Fig. 8.



These ordinates lie in the successive intervals between the ordinates M_0P_0, M_1P_1, ... M_nP_n; and it is easily shown that N_pQ_p (except where it is the maximum ordinate) is intermediate in magnitude between $M_{p-1}P_{p-1}$ and M_pP_p. Also we have

$$\alpha\,.\,N_pQ_p + \beta\,.\,N_{p+1}Q_{p+1} = \sqrt{n}\,.\,\alpha^p\beta^{n-p}\,(C^{n-1}_{p-1} + C^{n-1}_p) = \sqrt{n}\,.\,\alpha^p\beta^{n-p}C^n_p = M_pP_p.$$

But $N_pM_p : M_pN_{p+1} :: \beta : \alpha$; and therefore P_p lies in Q_pQ_{p+1}. It follows that, in the limit, Q_pQ_{p+1} becomes the tangent at P_p.

Let Q_pQ_{p+1} meet X′X in T_p. Then

$$\frac{M_pP_p}{M_pT_p} = \frac{N_pQ_p - N_{p+1}Q_{p+1}}{N_pN_{p+1}} = n\,.\,\alpha^{p-1}\,\beta^{n-p-1}\,\{\beta C^{n-1}_{p-1} - \alpha C^{n-1}_p\}$$
$$= \alpha^{p-1}\beta^{n-p-1}C^n_p\,\{p\beta - (n-p)\,\alpha\}.$$

Hence if we choose the point O so that

$$\sqrt{n}.OM_p = -n\alpha + p = p\beta - (n-p)\,\alpha,$$

we have

$$OM_p\,.\,M_pT_p = \alpha\beta.^{*}$$

* When n is not infinite, the relation $OM_p\,.\,M_pT_p = \alpha\beta$ shows that, if Σ denote any one of the family of normal curves of parameter $2\sqrt{\alpha\beta}$ having their median at O, the sides of the polygon $N_0Q_1Q_2$ . .

Now let n become indefinitely great, the point O remaining fixed. Then this relation holds all along the curve which is the limit of the polygon $P_0P_1 \ldots P_n$, and therefore this curve is a normal curve of parameter $2\sqrt{\alpha\beta}$, having its central ordinate at O. The mean value of α' is found by putting $OM = 0$, which gives $\alpha' = p/n = \alpha$. Thus the values of $\sqrt{n}\,(\alpha' - \alpha)$ are distributed normally with mean square $\alpha\beta = \alpha\,(1 - \alpha)$ about a mean value zero; and therefore the values of $\alpha' - \alpha$ are distributed normally with mean square $\alpha\,(1 - \alpha)/n$.

(ii.) Next, consider the distribution of values of $\alpha' - \alpha$ when certain other errors, as $\beta' - \beta$ and $\gamma' - \gamma$, have particular values. This distribution is found by taking an indefinitely great number of random selections, each containing n individuals, and isolating those sets in which the numbers drawn from the classes B and C are respectively $n\beta'$ and $n\gamma'$. From the principles of random selection it follows that the distribution of values of $\alpha' - \alpha$ in these sets is the same as if we made random selections of $n\,(1 - \beta' - \gamma')$ individuals from that portion of the community which does not involve B and C. Of this portion of the community, the class A forms a part denoted by the fraction $\alpha/(1 - \beta - \gamma)$. Hence the values of $n\alpha'$, the number coming from A, are distributed with mean square $n\,(1 - \beta' - \gamma') \times \alpha\,(1 - \alpha - \beta - \gamma)/(1 - \beta - \gamma)^2$ about a mean value $n\,(1 - \beta' - \gamma') \times \alpha/(1 - \beta - \gamma)$. So long as $\beta' - \beta$ and $\gamma' - \gamma$ are small in comparison with β and γ, this is equivalent to saying that the values of α' are distributed with constant mean square about a mean value $\alpha\,(1-\beta'-\gamma')/(1-\beta-\gamma)$ $= \alpha - \lambda\,(\beta' - \beta) - \lambda\,(\gamma' - \gamma)$, where $\lambda = \alpha/(1-\beta-\gamma)$. Thus the distributions of $\alpha' - \alpha$, $\beta' - \beta$, $\gamma' - \gamma, \ldots$ are normally correlated; and therefore, since the separate distributions are normal, the values of $a\,(\alpha' - \alpha) + b(\beta' - \beta) + c\,(\gamma' - \gamma) + \ldots$ are normally distributed.

Since this argument only applies when $\alpha' - \alpha$, $\beta' - \beta$, $\gamma' - \gamma, \ldots$ are small, the result is subject to the limitation pointed out in (1) (above).

§ 17. *Probable Error and Probable Discrepancy.*—Let X be any magnitude which is determined by observation of the ratios $\alpha', \beta', \gamma', \ldots$ Then X can be written in the form $f(\alpha', \beta', \gamma', \ldots)$. Now suppose n to be very great. Then the values of $\alpha' - \alpha$, $\beta' - \beta$, $\gamma' - \gamma, \ldots$ are distributed normally with mean values zero and mean squares $\alpha\,(1 - \alpha)/n$, $\beta\,(1 - \beta)/n$, $\gamma\,(1 - \gamma)/n, \ldots$; and therefore it may be supposed that in any particular case the values of $\alpha' - \alpha$, $\beta' - \beta$, $\gamma' - \gamma, \ldots$ will be very

Q_nN_{n+1} have the same slope at the points $P_1P_2 \ldots P_{n+1}$ as the respective curves Σ which pass through those points. Professor KARL PEARSON has arrived at a different result ('Phil. Trans.,' A, vol. 186 (1895) p. 357) by forming the polygon $P_1P_2 \ldots P_{n+1}$ and finding the "slope" at the middle points of its sides. There is of course no discrepancy between the two results, since they deal with different polygons, and with points having different relative positions on these polygons. The curve found by Professor PEARSON becomes the normal curve when n is made indefinitely great.

To prevent misunderstanding, it should be pointed out that, in either case, the slope of the polygon at the points in question is not the same as the slope of any *one* curve of the family considered. Professor PEARSON's statement (*op. cit.*, p. 356) as to the existence of a close relation between the binomial polygon (for $\alpha = \beta$) and "the" normal curve seems to require some qualification.

small. Thus X is of the form $f(\alpha,\beta,\gamma\ldots)+f_\alpha(\alpha'-\alpha)+f_\beta(\beta'-\beta)+f_\gamma(\gamma'-\gamma)+\ldots$; and therefore, by § 16, its mean value is $f(\alpha, \beta, \gamma, \ldots)$, and the different possible values are distributed normally about this mean value with mean square

$$\{(\alpha f_\alpha^2 + \beta f_\beta^2 + \gamma f_\gamma^2 + \ldots) - (\alpha f_\alpha + \beta f_\beta + \gamma f_\gamma + \ldots)^2\}/n.$$

If we denote the expression in curled brackets by σ^2, the quartile deviation from the mean is $Q\sigma/\sqrt{n}$, where Q is the deviation of the quartile ordinate from the central ordinate in the standard normal curve (= ·67449 approximately*).

The applications are of two kinds. In one class of cases X is a "frequency-constant" whose value is required. Its observed value $f'(\alpha', \beta', \gamma', \ldots)$ differs from its true value $f(\alpha, \beta, \gamma, \ldots)$ by an *error* due to the paucity of observations, and $Q\sigma/\sqrt{n}$ is then the *probable error*. In the other class of cases the theory is applied to the testing of any hypothesis with regard to numerical statistics. The difference between the observed and the calculated values of X is a *discrepancy*, and we test the hypothesis that this discrepancy is due to paucity of observations by comparing it with the *probable discrepancy* $Q\sigma/\sqrt{n}$. If the comparison is made for several different values of X, we ought to find that for about half of them the discrepancy $(= d)$ is less than the probable discrepancy $(= q)$, and that, amongst the remaining values, d is in no case a very large multiple of q. The following considerations will enable us to determine whether, in any particular case, the values of d/q are or are not greater than we might reasonably expect.

Let the different values of a magnitude δ be distributed normally, with quartile deviation q, about a mean value zero; and let m values be taken at random. Then, if the area of the standard normal figure lying between the ordinates at the points $x = -\rho/q$ and $x = +\rho/q$ is ϕ, the probability of one at least of the values of δ being numerically greater than ρ is $1 - \phi^m$. If we choose ϕ so that this probability may be equal to $\frac{1}{2}$, the corresponding value of ρ may, by analogy with the "probable error," be called the *probable limit* of δ. The following table gives the values of ρ/q determined by this condition, for values of m from 1 to 20† :—

m	ρ/q	m	ρ/q	m	ρ/q	m	ρ/q
1	1·000	6	2·375	11	2·777	16	3·009
2	1·559	7	2·481	12	2·832	17	3·046
3	1·874	8	2·570	13	2·882	18	3·080
4	2·088	9	2·648	14	2·928	19	3·112
5	2·248	10	2·716	15	2·970	20	3·142

* The value of Q to 20 places of decimals is ·67448 97501 96081 74320, and its logarithm to 13 places is $\bar{1}$·82897 53543 532. The successive convergents to Q are $\frac{1}{2}, \frac{2}{3}, \frac{27}{40}, \frac{29}{43}, \frac{201}{298}, \frac{230}{341}, \ldots$

† For larger values of m, the value of ρ/q may be taken as equal to that given by CHAUVENET'S criterion for the rejection of one out of $m/\log_e 4 + \frac{1}{4}$ observations.

If m values of X were observed, and if the discrepancies were independent, it would be an even chance that in one case at least the ratio of the discrepancy to the probable discrepancy would exceed the value given by the above table. As a matter of fact, the discrepancies are usually correlated; but, if we bear this in mind, the table may be used to decide whether the greatest value of the ratio is such as to negative the hypothesis under consideration.

For calculating $Q\sigma/\sqrt{n}$, in either class of cases, it will not always be necessary to express σ^2 in terms of $\alpha, \beta, \gamma, \ldots$ If the value of X depends solely on the values of certain frequency-constants, and if $\varsigma, \eta, \theta, \ldots$ are the errors in these frequency-constants, then $f(\alpha', \beta', \gamma', \ldots) - f(\alpha, \beta, \gamma, \ldots)$ may be written in the form $k\varsigma + l\eta + m\theta + \ldots$ The errors $\varsigma, \eta, \theta, \ldots$ being of the form $a(\alpha' - \alpha) + b(\beta' - \beta) + c(\gamma' - \gamma) + \ldots$, their mean squares and mean products can be found; and thence the mean square of $k\varsigma + l\eta + m\theta + \ldots$ can be obtained by the general formula given in § 12. The expressions for the mean squares and mean products of the errors in frequency-constants of certain particular forms will be found in §§ 18 and 19.

The true values of $\alpha, \beta, \gamma, \ldots$, or of the frequency-constants on which X depends, are not known; and therefore, in calculating $Q\sigma/\sqrt{n}$, we can only use the observed values $\alpha', \beta', \gamma', \ldots$ But, n being great, the mistake so introduced in $Q\sigma/\sqrt{n}$ is small in comparison with $Q\sigma/\sqrt{n}$ itself. In general, it is sufficient to determine $Q\sigma/\sqrt{n}$ within about 1 per cent. of its true value. It will therefore be found simplest to calculate σ^2/n, and then to take out the corresponding value of $Q\sigma/\sqrt{n}$ from Table V. (p. 159). This table gives $Q\sqrt{N}$, for any given value of N, within from ·8 to ·08 per cent. of its true value.

§ 18. *Error in Mean, Mean Square, &c.*—Let the mean value of a measure L (in an indefinitely great community), and the pth power of the deviation from the mean, be denoted by L_1 and λ_p respectively. Also let the actual values of L be $L_1 + x_1$, $L_1 + x_2$, $L_1 + x_3, \ldots$; and let the relative frequencies of these values be $z_1, z_2, z_3, \ldots$ Thus we have $\Sigma z = 1$, $\Sigma zx = 0$, $\Sigma zx^p = \lambda_p$. Now let a random selection of n individuals be made, and let the numbers for which L has the values $L_1 + x_1$, $L_1 + x_2$, $L_1 + x_3, \ldots$, be respectively $n(z_1 + \epsilon_1)$, $n(z_2 + \epsilon_2)$, $n(z_3 + \epsilon_3), \ldots$ Then (§§ 15, 16) the mean value of $A_1\epsilon_1 + A_2\epsilon_2 + A_3\epsilon_3 + \ldots \equiv \Sigma A\epsilon$ is zero; its mean square is $\{\Sigma A^2 z - (\Sigma Az)^2\}/n$; the mean product of $\Sigma A\epsilon$ and $B_1\epsilon_1 + B_2\epsilon_2 + B_3\epsilon_3 + \ldots \equiv \Sigma B\epsilon$ is $(\Sigma ABz - \Sigma Az . \Sigma Bz)/n$; and, n being supposed to be great, the values of $\Sigma A\epsilon$ or of $\Sigma B\epsilon$ are normally distributed.

Hence we obtain the following results:—

(i.) The calculated value of L_1 is $L_1 + (x_1\epsilon_1 + x_2\epsilon_2 + x_3\epsilon_3 + \ldots)$. Thus the error in L_1 is $x_1\epsilon_1 + x_2\epsilon_2 + x_3\epsilon_3 + \ldots$, and therefore this error is distributed normally with mean square $\{\Sigma zx^2 - (\Sigma zx)^2\}/n = \lambda_2/n$.

(ii.) Denote the error in L_1 by ω. Then the calculated value of λ_p is

$$\Sigma (z + \epsilon) (x - \omega)^p = \Sigma (z + \epsilon) (x^p - px^{p-1}\omega);$$

and therefore the error in λ_p is

$$\Sigma x^p \epsilon - p\Sigma z x^{p-1}\omega = \Sigma x^p \epsilon - p\lambda_{p-1}\omega = \Sigma (x^p - p\lambda_{p-1} x)\epsilon.$$

Hence this error is distributed normally with mean square

$$[\Sigma z(x^p - p\lambda_{p-1}x)^2 - \{\Sigma z(x^p - p\lambda_{p-1}x)\}^2]/n = (\lambda_{2p} - 2p\lambda_{p+1}\lambda_{p-1} + p^2\lambda_{p-1}^2\lambda_2 - \lambda_p^2)/n.$$

In particular, the mean square of the error in λ_2 is $(\lambda_4 - \lambda_2^2)/n$.

(iii.) The mean product of the errors in L_1 and in λ_p is

$$\{\Sigma zx (x^p - p\lambda_{p-1} x) - \Sigma zx . \Sigma z (x^p - p\lambda_{p-1} x)\}/n = (\lambda_{p+1} - p\lambda_{p-1}\lambda_2)/n.$$

In particular, the mean product of the errors in the mean and in the mean square of deviation is λ_3/n.

(iv.) The mean product of the errors in λ_p and in λ_q is

$$\{\Sigma z (x^p - p\lambda_{p-1} x)(x^q - q\lambda_{q-1} x) - \Sigma z (x^p - p\lambda_{p-1} x) . \Sigma z (x^q - q\lambda_{q-1} x)\}/n$$
$$= (\lambda_{p+q} - p\lambda_{p-1}\lambda_{q+1} - q\lambda_{p+1}\lambda_{q-1} + pq\lambda_{p-1}\lambda_{q-1}\lambda_2 - \lambda_p\lambda_q)/n.$$

§ 19. *Error in Class-Index.*—Let the values $L_1 + x_1$, $L_1 + x_2$, $L_1 + x_3 \ldots$, in § 18, be supposed to be in order of magnitude, $L_1 + x_1$ being least; and let X be any possible value of L, not coinciding with any one of these actual values.* Let the two classes for which L is respectively less and greater than X be denoted by C′ and C, and let the numbers in these classes be in the ratio of $1 + \alpha : 1 - \alpha$; then α will be called the *class-index* of X for classification according to values of L. Its value ranges from -1 to $+1$.

If a representative selection of n individuals were made, the numbers coming from the two classes would be $n_1 = \frac{1}{2} n (1 + \alpha)$ and $n_2 = \frac{1}{2} n (1 - \alpha)$; so that $\alpha = (n_1 - n_2)/(n_1 + n_2)$. Suppose however that the selection is a random one, the errors being as in § 18. Then, if we take X as lying between X_r and X_{r+1}, the observed value of α is $(z_1 + \epsilon_1) + (z_2 + \epsilon_2) + \ldots + (z_r + \epsilon_r) - (z_{r+1} + \epsilon_{r+1}) - \ldots$, and therefore the "error" in α is $\epsilon_1 + \epsilon_2 + \ldots + \epsilon_r - \epsilon_{r+1} - \ldots$ Hence:—

(i.) By considering the division of the community into the two classes C′ and C, we see from § 15 (i.) and (ii.) that the error in α is distributed normally with mean square $(1 - \alpha^2)/n$ about a mean value zero.

* This limitation does not introduce any difficulty in the case of continuous variation, since the frequency of any single value is then indefinitely small. (Cases in which the curve of frequency has an infinite ordinate are excluded from consideration.)

(ii.) Let β be another class-index. The lines of division corresponding to these two class-indices divide the community into three classes, whose numbers are proportional to quantities Z_1, Z_2, Z_3, where $Z_1 + Z_2 + Z_3 = 1$. From § 15 (iii.) it will be seen that the mean product of the errors in α and in β is

$$4Z_1Z_3/n = \{(1 - \alpha\beta) - (\alpha \backsim \beta)\}/n.$$

(iii.) Let the values of Σzx^p for the classes C′ and C be respectively ν'_p and ν_p, so that $\nu_p + \nu'_p = \lambda_p$. Then it will be found from § 15 (vi.) that the mean product of the errors in α and in L_1 is $-(\nu_1 - \nu'_1)/n$; and that the mean product of the errors in α and in λ_p is

$$-\{(\nu_p - \nu'_p) - (\nu_1 - \nu'_1)\,p\lambda_{p-1} + \alpha\lambda_p\}/n.$$

The following table shows the general results obtained in this and the last section; for convenience, the divisor n is omitted throughout.

	L_1	λ_p	α
L_1	λ_2	$\lambda_{p+1} - p\lambda_{p-1}\lambda_2$	$-(\nu_1 - \nu'_1)$
λ_p		$\lambda_{2p} - 2p\lambda_{p+1}\lambda_{p-1} + p^2\lambda^2_{p-1}\lambda_2 - \lambda^2_p$	$-\{(\nu_p - \nu'_p) - (\nu_1 - \nu'_1)\,p\lambda_{p-1} + \alpha\lambda_p\}$
λ_q		$\lambda_{p+q} - p\lambda_{p-1}\lambda_{q+1} - q\lambda_{p+1}\lambda_{q-1} + pq\lambda_{p-1}\lambda_{q-1}\lambda_2 - \lambda_p\lambda_q$	(Similar expression)
α			$1 - \alpha^2$
β			$(1 - \alpha\beta) - (\alpha \backsim \beta)$

§ 20. *Mean Squares and Products of Errors in Case of Two Attributes.*—Let M be the measure of a second attribute, M_1 its mean value, and μ_q the mean qth power of the deviation from the mean; and suppose that each z in § 18 denotes the proportion of individuals for which L and M jointly have certain specified values. Let $S_{p,q}$ denote the mean value of $(L - L_1)^p\,(M - M_1)^q$, so that $S_{p,0} = \lambda_p$, $S_{0,q} = \mu_q$. Then it will be found that the error in $S_{p,q}$ (*i.e.*, the error produced by taking $S_{p,q}$ as equal to the average of $x^p\,y^q$, where x and y are the respective deviations of L and M from their averages for the n individuals) is of the form $\Sigma A\epsilon$, and therefore is distributed normally; its mean square being

$$[\Sigma z\,(x^py^q - pS_{p-1,q}x - qS_{p,q-1}y)^2 - \{\Sigma z\,(x^py^q - pS_{p-1,q}x - qS_{p,q-1}y)\}^2]/n$$
$$= (S_{2p,2q} - 2pS_{p+1,q}S_{p-1,q} - 2qS_{p,q+1}S_{p,q-1} + p^2S^2_{p-1,q}\lambda_2$$
$$+ 2pqS_{p-1,q}S_{p,q-1}S_{1,1} + q^2S^2_{p,q-1}\mu_2 - S^2_{p,q})/n.$$

Let X and Y be the values of L and M corresponding to class-indices α and β; and let $\frac{1}{2}(1-\chi)$ be the proportion of individuals for which L exceeds X and M exceeds Y: thus χ is necessarily greater than either α or β. Let the constituent parts of $S_{p,q}$ corresponding to $\frac{1}{2}(1-\chi)$ and $\frac{1}{2}(1+\chi)$ be $\sigma_{p,q}$ and σ'_{pq} respectively, so that, if a representative selection of N individuals is made, the value of $\Sigma(L-L_1)^p(M-M_1)^q$ for the $\frac{1}{2}N(1-\chi)$ individuals for which L exceeds X and M exceeds Y is $N\sigma_{p,q}$, while for the remaining $\frac{1}{2}N(1+\chi)$ it is $N\sigma'_{p,q}$. Then it can be shown by the methods of §§ 18 and 19 that the following tables give the mean products of the errors in the quantities concerned, the divisor n being omitted:—

	L_1	λ_l	$S_{p,q}$
M_1	$S_{1,1}$	$S_{l,1}-l\lambda_{l-1}S_{1,1}$	$S_{p,q+1}-pS_{p-1,q}S_{1,1}-qS_{p,q-1}\mu_2$
μ_m	$S_{1,m}-m\mu_{m-1}S_{1,1}$	$S_{l,m}-l\lambda_{l-1}S_{1,m}-mS_{l,1}\mu_{m-1}$ $+lm\lambda_{l-1}\mu_{m-1}S_{1,1}-\lambda_l\mu_m$	$S_{p,q+m}-pS_{p-1,q}S_{1,m}-q\mu_{m+1}S_{p,q-1}$ $-m\mu_{m-1}S_{p,q+1}+mp\mu_{m-1}S_{p-1,q}S_{1,1}$ $+mq\mu_{m-1}S_{p,q-1}\mu_2-\mu_mS_{p,q}$
$S_{r,s}$	$S_{r+1,s}-rS_{r-1,s}\lambda_2$ $-sS_{r,s-1}S_{1,1}$	$S_{l+r,s}-l\lambda_{l-1}S_{r+1,s}-r\lambda_{l+1}S_{r-1,s}$ $-sS_{l,1}S_{r,s-1}+lr\lambda_{l-1}S_{r-1,s}\lambda_2$ $+ls\lambda_{l-1}S_{r,s-1}S_{1,1}-\lambda_lS_{r,s}$	$S_{p+r,q+s}-pS_{p-1,q}S_{r+1,s}-qS_{p,q-1}S_{r,s+1}$ $-rS_{p+1,q}S_{r-1,s}-sS_{p,q+1}S_{r,s-1}$ $+prS_{p-1,q}S_{r-1,s}\lambda_2+qrS_{p,q-1}S_{r-1,s}S_{1,1}$ $+psS_{p-1,q}S_{r,s-1}S_{1,1}+qsS_{p,q-1}S_{r,s-1}\mu_2$ $-S_{p,q}S_{r,s}$
χ	$-(\sigma_{1,0}-\sigma'_{1,0})$	$-\{(\sigma_{l,0}-\sigma'_{l,0})-l\lambda_{l-1}(\sigma_{1,0}-\sigma'_{1,0})+\chi\lambda_l\}$	$-\{(\sigma_{p,q}-\sigma'_{p,q})-pS_{p-1,q}(\sigma_{1,0}-\sigma'_{1,0})$ $-qS_{p,q-1}(\sigma_{0,1}-\sigma'_{0,1})+\chi S_{p,q}\}$
α, β	(similar expressions)		

	χ	α	β
χ	$1-\chi^2$	$(1-\chi)(1+\alpha)$	$(1-\chi)(1+\beta)$
α		$1-\alpha^2$	$1+\alpha+\beta-\alpha\beta-2\chi$
β			$1-\beta^2$

Suppose, for instance, that we are considering the error in $S_{1,1}/\sqrt{\lambda_2\mu_2}\equiv k$. Let the errors in λ_2, in $S_{1,1}$, and in μ_2 be θ, ϕ, and ψ respectively; then the error in k is $(-\theta/2\lambda_2 + \phi/S_{1,1} - \psi/2\mu_2)\,k$. For the mean squares and mean products of θ, ϕ, ψ, we have the table—

	λ_2	$S_{1,1}$	μ_2
λ_2	$\lambda_4 - \lambda_2^2$	$S_{3,1} - \lambda_2 S_{1,1}$	$S_{2,2} - \lambda_2\mu_2$
$S_{1,1}$		$S_{2,2} - S_{1,1}^2$	$S_{1,3} - \mu_2 S_{1,1}$
μ_2			$\mu_4 - \mu_2^2$

from which it will be found that the mean square of the error in k is

$$\left\{\frac{\lambda_4}{4\lambda_2^2} + \left(\frac{1}{S_{1,1}^2} + \frac{1}{2\lambda_2\mu_2}\right)S_{2,2} + \frac{\mu_4}{4\mu_2^2} - \frac{S_{3,1}}{\lambda_2 S_{1,1}} - \frac{S_{1,3}}{\mu_2 S_{1,1}}\right\}\Big/ n.$$

§ 21.—*Test of Independence of Two Distributions.*—For an illustration of the application of the theory of error to testing statistical hypotheses, let us take the case of two independent distributions. The criterion of independence of the distributions of two measures L and M is that, if α denotes the proportion of individuals, in the complete community, for which L lies between any two values L′ and L″, and if β denotes the proportion for which M lies between any two values M′ and M″, then the proportion for which both these conditions are satisfied is $\alpha\beta$. Hence, in order to test the hypothesis of independence when n individuals have been obtained by random selection, we must arrange them in a table of double entry, thus:—

Values of L.	Values of M.			Total.
	M′ to M″.	M″ to M‴.	&c.	
L′ to L″	n_{11}	n_{12}	&c.	p_1
L″ to L‴	n_{21}	n_{22}		p_2
⋮	⋮	⋮		⋮
Total . . .	q_1	q_2	. . . . .	n

then form a new table by dividing each number in this table by n, so as to show the proportions in the different classes; and then consider whether the discrepancies between these proportions and the corresponding proportions in a table showing independent distribution are such as might be accounted for by random selection.

Let the following table represent the proportions, in the original community, of the individuals specified :—

Values of L.	Values of M.	
	M″ to M‴.	Remainder.
L′ to L″	V	V″
Remainder	V′	V‴

and let ψ, ψ', ψ'', ψ''' be the errors in V, V′, V″, V‴. Thus $n_{12} = n(\mathrm{V} + \psi)$, $p_1 = n(\mathrm{V} + \mathrm{V}'' + \psi + \psi'')$, $q_2 = n(\mathrm{V} + \mathrm{V}' + \psi + \psi')$. If the distributions are independent, $\mathrm{V} = (\mathrm{V}+\mathrm{V}')(\mathrm{V}+\mathrm{V}'')$; *i.e.* (since $\mathrm{V}+\mathrm{V}'+\mathrm{V}''+\mathrm{V}'''=1$), $\mathrm{VV}'''=\mathrm{V}'\mathrm{V}''$. Hence (since $\psi + \psi' + \psi'' + \psi''' = 0$)

$$n_{12} - p_1 q_2/n = n\{\psi - (\mathrm{V} + \mathrm{V}')(\psi + \psi'') - (\mathrm{V} + \mathrm{V}'')(\psi + \psi')\}$$
$$= n\{(\mathrm{V}'''\psi + \mathrm{V}\psi''') - (\mathrm{V}''\psi' + \mathrm{V}'\psi'')\}.$$

By § 15 (v.) it will be found that the mean square of this discrepancy is $n\mathrm{VV}''' = n\mathrm{V}'\mathrm{V}''$; and therefore the "probable discrepancy" is $\mathrm{Q}\sqrt{n\mathrm{VV}'''} = \mathrm{Q}\sqrt{n\mathrm{V}'\mathrm{V}''}$. By calculating this expression for each number in the table, and comparing the actual discrepancies, as $n_{12} - p_1 q_2/n$, with the values so obtained, we have data for deciding as to the validity of the hypothesis of independence.

The following example of a case in which, on *a priori* grounds, we should expect to find independence, will serve as an illustration. The table is compiled from a list of school-teachers who passed a certain examination.

List.	First letter of name.				Total.
	A–D.	E–J.	K–R.	S–Z.	
Men	166	174	180	164	684
Women, 1st year	427	379	411	366	1583
,, 2nd ,, ...	549	493	577	492	2111
Total	1142	1046	1168	1022	4378

By multiplying each total of a row by each total of a column, and dividing each product by $n = 4378$, we get the "calculated" table

178·4	163·4	182·5	159·7
413·0	378·2	422·3	369·5
550·6	504·4	563·2	492·8

showing discrepancies in the actual table amounting to

− 12·4	+ 10·6	− 2·5	+ 4·3
+ 14·0	+ 0·8	− 11·3	− 3·5
− 1·6	− 11·4	+ 13·8	− 0·8

If nV represents any number in the calculated table, the corresponding values of nV‴ will be found to be

2730·4	2811·4	2708·5	2831·7
2066·0	2127·2	2049·3	2142·5
1675·6	1725·4	1662·2	1737·8

Multiplying each number in this table by the corresponding number in the "calculated" table, and dividing by 4378, we get the values of nVV‴

111·26	104·93	112·91	103·29
194·90	183·76	197·67	180·83
210·73	198·79	213·83	195·61

Whence, from Table V. (p. 159) the probable discrepancies are

7·1	6·9	7·2	6·9
9·4	9·1	9·5	9·1
9·8	9·5	9·9	9·4

The ratios of the actual discrepancies to these probable discrepancies are

− 1·7	+ 1·5	− 0·3	+ 0·6
+ 1·5	+ 0·1	− 1·2	− 0·4
− 0·2	− 1·2	+ 1·4	− 0·1

Thus six out of the twelve ratios are numerically less than unity, and six numerically greater, while the greatest ratio is well within the probable limit (§ 17). The hypothesis of independence in this case is therefore justified by the data.*

Part III.—Application to Normal Distributions.

§ 22. *Probable Errors in Mean and in Semi-parameter by Different Methods.*—If the values of a measure L are known to be distributed normally, the distribution is

* The method of this section is an extension of the ordinary method (used largely by Professor Lexis and Professor Edgeworth) for testing the "stability of statistical ratios."

determined when the mean value L_1 and the semi-parameter a are determined. When the values of L for n individuals obtained by random selection are given, the values of L_1 and of a can be found in either of two different ways.

(1.) We can find the average and the standard deviation (square root of average square of deviation from the average*) of the n individuals. The average will differ from L_1 by an error whose mean square (§ 18 (i.)) is a^2/n, so that the probable error of L_1 as found in this way is $Qa/\sqrt{n}$; and (§ 18 (ii.)) the square of the standard deviation will differ from a^2 by an error whose mean square is $(\lambda_4 - \lambda_2^2)/n = 2a^2/n$ (§ 5); so that the probable error in a will be $Qa/\sqrt{2n}$. These are familiar results.

(2.) The other method is that which has been mainly used by Mr. GALTON.† Let α and β be any two class-indices, and let X and Y be the corresponding values of L in the complete community. Then, if x and y are the abscissæ corresponding to class-indices α and β in the standard normal figure (*i.e.*, if ordinates at distances x and y from the central ordinate divide the figure into areas whose ratios are $1 + \alpha : 1 - \alpha$ and $1 + \beta : 1 - \beta$ respectively), we have

$$\left.\begin{aligned} X &= L_1 + ax \\ Y &= L_1 + ay \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (i.).$$

Whence

$$\left.\begin{aligned} L_1 &= (xY - yX)/(x - y) \\ a &= (X - Y)/(x - y) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (ii.).$$

Now let ξ and η be the errors in the observed values of X and of Y; *i.e.*, let α and β be the class-indices of $X + \xi$ and $Y + \eta$ in the collection of n individuals. Then, if we deduce the values of L_1 and of a from (ii.), the resulting errors are $-(y\xi - x\eta)/(x - y)$ and $(\xi - \eta)/(x - y)$ respectively. Now the errors ξ and η are due to errors $-2z\xi/a$ and $-2z'\eta/a$ in the class-indices of X and Y, where z and z' are the ordinates of the standard normal figure corresponding to abscissæ x and y; and therefore (§ 19) the mean squares and mean product of ξ and η are $a^2(1 - \alpha^2)/4nz^2$, $a^2(1 - \beta^2)/4nz'^2$, and $a^2\{(1 - \alpha\beta) - (\alpha \sim \beta)\}/4nzz'$. Hence the probable errors in L_1 and in a, as found from (ii.), are respectively $Q.E/\sqrt{n}$ and $Q.H/\sqrt{n}$, where

$$\left.\begin{aligned} E^2 &= a^2\left\{\left(\frac{y}{x-y}\right)^2\frac{1-\alpha^2}{4z^2} - \frac{2xy}{(x-y)^2}\,\frac{(1-\alpha\beta)-(\alpha\sim\beta)}{4zz'} + \left(\frac{x}{x-y}\right)^2\frac{1-\beta^2}{4z'^2}\right\} \\ H^2 &= \frac{a^2}{(x-y)^2}\left\{\frac{1-\alpha^2}{4z^2} - 2\,\frac{(1-\alpha\beta)-(\alpha\sim\beta)}{4zz'} + \frac{1-\beta^2}{4z'^2}\right\} \end{aligned}\right\} \quad . \quad (iii.).$$

* It seems convenient to use the term "standard deviation" in this sense, as denoting a quantity which has a definite value for the particular data.

† GALTON, 'Natural Inheritance,' p. 62.

(3.) As an extension of this last result, let X, Y, U, ... be values of L corresponding (in the complete community) to class-indices $\alpha, \beta, \gamma, \ldots$, and let the corresponding abscissæ in the standard figure be $x, y, u, \ldots$ Then $X = L_1 + ax$, $Y = L_1 + ay$, $U = L_1 + au, \ldots$; and therefore

$$\left.\begin{aligned} L_1 &= (lX + mY + pU + \ldots)/(l + m + p + \ldots) \\ a &= (l'X + m'Y + p'U + \ldots)/(l'x + m'y + p'u + \ldots) \end{aligned}\right\} \quad \text{(i.)},$$

where $l, m, p \ldots, l', m', p' \ldots$ are any quantities which satisfy the conditions

$$\left.\begin{aligned} lx + my + pu + \ldots &= 0 \\ l' + m' + p' + \ldots &= 0 \end{aligned}\right\} \quad \text{(ii.)}.$$

Suppose that we fix on the values of $\alpha, \beta, \gamma, \ldots$ beforehand, and choose $l, m, p, \ldots, l', m', p', \ldots$ to satisfy (ii.), and then observe the values of L whose class-indices in the collection of n individuals are $\alpha, \beta, \gamma, \ldots$ If the errors in these values are $\xi, \eta, \theta, \ldots$, the resulting errors in L_1 and in a will be $(l\xi + m\eta + p\theta + \ldots)/(l + m + p + \ldots)$ and $(l'\xi + m'\eta + p'\theta + \ldots)/(l'x + m'y + p'u + \ldots)$; and therefore the probable errors in L_1 and in a, as deduced from (i.), are $Q.E/\sqrt{n}$ and $Q.H/\sqrt{n}$, where

$$\left.\begin{aligned} E^2 &= \tfrac{1}{4}a^2 \left\{ \frac{l^2(1-\alpha^2)}{z^2} + \frac{m^2(1-\beta^2)}{z'^2} + \ldots \right. \\ &\qquad \left. + \frac{2lm\{(1-\alpha\beta) - (\alpha \backsim \beta)\}}{zz'} + \ldots \right\} \Big/ (l + m + \ldots)^2 \\ &= \tfrac{1}{4}a^2 \left\{ \left(\Sigma \frac{l}{z}\right)^2 - \left(\Sigma \frac{l\alpha}{z}\right)^2 - 2\Sigma \frac{lm(\alpha \backsim \beta)}{zz'} \right\} \Big/ (\Sigma l)^2 \\ H^2 &= \tfrac{1}{4}a^2 \left\{ \left(\Sigma \frac{l'}{z}\right)^2 - \left(\Sigma \frac{l'\alpha}{z}\right)^2 - 2\Sigma \frac{l'm'(\alpha \backsim \beta)}{zz'} \right\} \Big/ (\Sigma l'x)^2 \end{aligned}\right\} \quad \text{(iii.)}.$$

For any particular values of $\alpha, \beta, \gamma, \ldots$, the values of $l, m, p, \ldots, l', m', p', \ldots$ can be chosen so as to reduce E^2 or H^2 to a minimum.

§ 23. *Relative Accuracy of the Different Methods.*—Now let ω and ρ be the errors in L_1 and in a as obtained by the *average-and-average-square* method; *i.e.*, the errors due to taking them as equal to the average and the standard deviation of the n individuals. Also let the class-index of X, in the n individuals, be $\alpha + \theta$, the true class-index of X being α. Then, with the notation of § 19, the mean values of $\omega\theta$ and of $2a\rho\theta$ are respectively $-(\nu_1 - \nu'_1)/n$ and $-(\nu_2 - \nu'_2 + a^2\alpha)/n$. But, by § 5, $\nu_1 = az$, $\nu'_1 = -az$, $\nu_2 = \frac{1}{2}(1-\alpha)a^2 + a^2xz$, $\nu'_2 = \frac{1}{2}(1+\alpha)a^2 - a^2xz$. Also the error ξ in X is due to the error θ, and is equal to $-a\theta/2z$. Thus we have the following table of mean squares and mean products of errors, the divisor n, as usual, being omitted :—

	L_1	a	X
L_1	a^2	0	a^2
a	0	$\frac{1}{2}a^2$	$\frac{1}{2}a^2x$
X	a^2	$\frac{1}{2}a^2x$	$a^2(1-\alpha^2)/4z^2$

and thence

	L_1	a	$X-(L_1+ax)$
L_1	a^2	0	0
a	0	$\frac{1}{2}a^2$	0
$X-(L_1+ax)$	0	0	$a^2(1-\alpha^2)/4z^2 - a^2 - \frac{1}{2}a^2x^2$

The true value of $X-(L_1+ax)$, of course, is zero; so that the "error" in $X-(L_1+ax)$ is the difference between X as determined by direct observation of the value whose class-index is α, and L_1+ax, as determined by calculating the average and the standard deviation. This error is $\xi-(\omega+x\rho)$; and therefore, if we write $\xi=\omega+x\rho+\phi$, the last table shows that the mean products of ω, ρ, and ϕ, taken in pairs, are zero. Hence we deduce the following conclusions:—

(1.) The mean square of ξ is greater than the mean square of $\omega+x\rho$.* Hence, if we fix a class-index α, corresponding to abscissa x in the standard normal figure, and if X denote the unknown value of L whose class-index is α, the probable error in X as obtained by direct observation is greater† than the probable error in the value obtained by calculating the average and the standard deviation, and deducing X from the formula $X=L_1+ax$. The following table, for instance, gives the probable errors in certain values which are often chosen for exhibiting the frequency-constants in any particular case:—

* This shows that $a^2(1-\alpha^2)/4z^2 > a^2(1+\frac{1}{2}x^2)$. Hence, if OH is the central ordinate, and MP any other ordinate, of a normal figure of parameter $2a$, and if A_1 and A_2 are the areas into which the figure is divided by MP, the product A_1A_2 is greater than $MP^2(a^2+\frac{1}{2}OM^2)$.

† The result, of course, only holds when we *know* that the distribution is normal. When we know nothing about it, the value corresponding to any particular class-index can only be obtained by direct observation.

Value of L.	Value of α.	Probable error in L by direct observation.	Probable error by average-and-average-square method.	Ratio of probable errors.
Median	·0	$\cdot 84535\ a/\sqrt{n}$	$\cdot 67449\ a/\sqrt{n}$	1·25
Quartiles	$\pm$ ·5	$\cdot 91908\ a/\sqrt{n}$	$\cdot 74728\ a/\sqrt{n}$	1·23
Deciles	$\pm$ ·2	$\cdot 85528\ a/\sqrt{n}$	$\cdot 68523\ a/\sqrt{n}$	1·25
	$\pm$ ·4	$\cdot 88897\ a/\sqrt{n}$	$\cdot 71937\ a/\sqrt{n}$	1·24
	$\pm$ ·6	$\cdot 96369\ a/\sqrt{n}$	$\cdot 78489\ a/\sqrt{n}$	1·23
	$\pm$ ·8	$1\cdot 15298\ a/\sqrt{n}$	$\cdot 91023\ a/\sqrt{n}$	1·27

(2.) If we take L_1 as equal to the average for the n individuals, and find X and Y by observing the values of L whose class-indices are α and β respectively, the mean square of the resulting error in $L_1 - (x\,Y - y\,X)/(x - y)$ is

$$a^2/n - 2\,(x\,a^2/n - y\,a^2/n)/(x - y) + E^2/n = (E^2 - a^2)/n,$$

where E^2 has the value given in § 22 (2.); and similarly, if we take a as equal to the standard deviation of the n individuals, the mean square of the error in $a - (Y - X)/(x - y)$ is $(H^2 - \frac{1}{2}a^2)/n$. Hence E^2 and H^2 are respectively greater than a^2 and $\frac{1}{2}a^2$; in other words, the probable errors in the values of L_1 and of a as determined by the formulæ (ii.) of § 22 (2.), are greater than the probable errors in their values as determined by the average-and-average-square method of § 22 (1.).

If, for instance, $\alpha = -\beta = \pm\frac{1}{2}$, so that the observed values are the two quartiles, the probable error in L_1 as determined by (ii.) of § 22 (2.) is $\cdot 75043\ a/\sqrt{n}$, which is 11 per cent. greater* than the probable error $\cdot 67449\ a/\sqrt{n}$ due to the average-and-average-square method; and the probable error in a is $\cdot 78672\ a/\sqrt{n}$, which is nearly 65 per cent. greater than the probable error $\cdot 47694\ a/\sqrt{n}$ due to the average-and-average-square method.

If we are unable to calculate the average and the standard deviation, we should

* When the quartiles are observed, it is also usual to observe the "median," for which $\alpha = 0$. If we take the arithmetic mean of the median and the two quartiles, the probable error due to taking this as the value of L_1 is reduced to $\cdot 72736\ a/\sqrt{n}$, which is less than 8 per cent. in excess of the probable error due to taking the average. If X and Y are the quartiles and M the median, it may be shown that the best result from these data is obtained by giving to $\frac{1}{2}(X + Y)$ and M weights in the ratio of $2\,(\exp. -\frac{1}{2}Q^2) - 1 : (\exp. \frac{1}{2}Q^2) - 1$, and the probable error in the mean is then $[\frac{1}{2}Qa\sqrt{\pi}/\{1 - 2\,(\exp. -\frac{1}{2}Q^2) + 2\,(\exp. -Q^2)\}^{\frac{1}{2}}]/\sqrt{n}$. The first two convergents to the above ratio are 2 : 1 and 7 : 3, so that $\{7\,(X + Y) + 6\,M\}/20$ is a slightly better value than $(X + Y + M)/3$. I have assumed that the quartiles, &c., are found by actual observation. But there is reason to believe that their values are sometimes obtained by faulty methods of interpolation. This does not affect the magnitude of the probable error, but it affects the calculated values of L_1 and of a.

choose α and β so as to make the values of E^2 and of H^2 as small as possible. It is obvious that one of the class-indices must be positive and the other negative. Suppose α to be negative, and equal to $-\gamma$; then it will be found from KRAMP's tables that E^2 is a minimum when β and γ are each taken a little greater than ·459, the probable error in the mean being then ·74951 $a/\sqrt{n}$, which is about the same as the probable error due to using the quartiles; and that H^2 is a minimum when β and γ are each taken a little less than ·862, the probable error in the semi-parameter being then ·59055 $a/\sqrt{n}$, which is about 25 per cent. less than the probable error due to using the quartiles, but nearly 24 per cent. greater than that due to the average-and-average-square method.

(3.) Suppose the values of the mean and of the semi-parameter to be found by the extended class-index method of § 22 (3.). Then, with the notation used above, the errors in the observed values of X, Y, U, . . . are of the form $\omega + x\rho + \phi$, $\omega + y\rho + \psi$, $\omega + u\rho + \chi$, . . . where ϕ, ψ, χ, . . . are errors whose mean products with ω, and also with ρ, are zero. Substituting in (i.) of § 22 (3.), and taking account of (ii.), we see that the resulting errors in L_1 and in a due to this method are respectively

$$\omega + (l\phi + m\psi + p\chi + \ldots)/(l + m + p + \ldots)$$

and

$$\rho + (l'\phi + m'\psi + p'\chi + \ldots)/(l'x + m'y + p'u + \ldots).$$

Hence if Φ^2/n and Φ'^2/n are the mean squares of

$$(l\phi + m\psi' + p\chi + \ldots)/(l + m + p + \ldots)$$

and of

$$(l'\phi + m'\psi + p'\chi + \ldots)/(l'x + m'y + p'u + \ldots),$$

the mean squares of the errors in L_1 and in a, due to the use of the class-index method, are $(a^2 + \Phi^2)/n$ and $(\frac{1}{2}\, a^2 + \Phi'^2)/n$. Since these are necessarily greater than a^2/n and $\frac{1}{2}\, a^2/n$ respectively, the probable errors in L_1 and in a due to this method are greater than the probable errors due to the average-and-average-square method. In other words, we cannot, by observation of the values corresponding to particular class-indices, obtain such good results for L_1 and a as by calculating the average and the standard deviation.*

(4.) Generally, let R be any quantity which would be known if the true mean and mean square of the distribution were known; let R_1 be the value obtained by taking the mean and mean square as equal to the average and the average square for the n observations, and let R_2 be the value obtained by any other method involving observation of the class-indices of any finite number of values of L, with or without the

* Professor EDGEWORTH's contrary statement ('Phil. Mag.,' vol. 36, 1893, p. 100) appears to be based on neglect of the correlation of errors.

use of the average and the average square. Let Θ_1^2/n and Θ_2^2/n be the mean squares of the errors in R as determined by the two methods. Then it may be shown that the mean square of the error involved in taking $pR_1 + qR_2$ as the value of $(p+q)$ R is $\{(p^2 + 2pq)\Theta_1^2 + q^2\Theta_2^2\}/n = \{(p+q)^2\Theta_1^2 + q^2(\Theta_2^2 - \Theta_1^2)\}/n$. Since this must be positive, it follows, by taking $p + q = 0$, that Θ_2^2 must be greater than Θ_1^2; and therefore R_1 gives a better value of R than R_2. By taking $p = -q = \pm 1$ we see that the quartile of $R_1 \backsim R_2$ is $Q(\Theta_2^2 - \Theta_1^2)^{\frac{1}{2}}/\sqrt{n}$.

§ 24. *Test of Hypothesis as to Normal Distribution.*—To test whether any particular distribution is normal, we use the result obtained at the beginning of the last section. Having found the average and the standard deviation of the n individuals, we calculate $L_1 + ax$, the value which should correspond to class-index α. The difference between this and the observed value X is a discrepancy whose mean square is $a^2\{(1-\alpha^2)/4z^2 - (1 + \frac{1}{2}x^2)\}/n$, so that the probable discrepancy is $Qa\{(1-\alpha^2)/4z^2 - (1+\frac{1}{2}x^2)\}^{\frac{1}{2}}/\sqrt{n}$; and the actual discrepancy has to be compared, for as many values of x as possible, with this probable discrepancy.

Suppose, for instance, that we take the chest-measurements of Scotch soldiers,* to which QUETELET refers in the work quoted above :—

CHEST-MEASUREMENTS, to the nearest inch, of 5,732 Scotch soldiers.

Inches.	Number.	Inches.	Number.
33	3	41	935
34	19	42	646
35	81	43	313
36	189	44	168
37	409	45	50
38	753	46	18
39	1062	47	3
40	1082	48	1

The values of the average and of the standard deviation cannot, of course, be calculated exactly; as the most probable values we find† $L_1 = 39{\cdot}8489$ inches, $a = 2{\cdot}05301$ inches. Thus we get the following results :—

* 'Edinburgh Medical Journal,' vol. 13, pp. 260–262. QUETELET made some mistakes, which I have corrected, in transcribing the figures.

† The formula for calculating the standard deviation has been given by me in a paper "On the Calculation of the most Probable Values of Frequency-Constants," in vol. 29 of the 'Proceedings of the London Mathematical Society' (p. 353).

The values given in the text are obtained by a first approximation. A second approximation might be made by assuming that the data represent the result of a random selection from the normal distribution given by the first approximation; but this correction would not alter any discrepancy shown in the table by as much as 1 per cent., and it may therefore be omitted.

Value of L.	α.	x*.	$L_1 + ax$.	Discrepancy.	Probable discrepancy.	Ratio of actual to probable discrepancy.
32·5	−1·00000					
33·5	−0·99895					
34·5	−0·99232					
35·5	−0·96406	−2·09762	35·5425	−·0425	·0442	0·96
36·5	−0·89812	−1·63579	36·4906	+·0094	·0263	0·36
37·5	−0·75541	−1·16359	37·4600	+·0400	·0177	2·26
38·5	−0·49267	−0·66301	38·4877	+·0123	·0145	0·85
39·5	−0·12212	−0·15366	39·5334	−·0334	·0138	2·42
40·5	+0·25541	+0·32578	40·5177	−·0177	·0139	1·27
41·5	+0·58165	+0·80928	41·5104	−·0104	·0150	0·69
42·5	+0·80705	+1·30190	42·5217	−·0217	·0195	1·11
43·5	+0·91626	+1·72938	43·3993	+·1007	·0290	3·47
44·5	+0·97488	+2·23952	44·4467	+·0533	·0525	1·02
45·5	+0·99232					
46·5	+0·99860					
47·5	+0·99965					
48·5	+1·00000					

The extremities of the range are not considered, as the values of $\frac{1}{2}n(1+\alpha)$ or $\frac{1}{2}n(1-\alpha)$ are small when α is nearly equal to ± 1, so that the law of normal distribution does not hold with regard to the errors in these values; and, moreover, z is changing rapidly, so that ξ is not exactly proportional to θ. For the ten values considered, the actual discrepancy is less than the probable discrepancy in four cases, and greater in six; and for nine of them the ratio of the two is within the probable limit (§ 17). The remaining ratio is rather large (3·47); but otherwise the data appear to justify the hypothesis of normal distribution.†

* The values of x shown in this column correspond to the fractional values of α given by the data (− 2763/2866, − 2574/2866, &c.), not to the nearest decimal values as shown in the second column (− ·96406, − ·89812, &c.).

The quantities shown in the final column are the ratios of the quantities given in the preceding columns. If these were taken to the fifth place of decimals, the last figure in some of the ratios might be altered; but it is not necessary to make such exact calculations (§ 17).

† It should be remembered that when the probable discrepancy is small, the possibility of errors of scale must be considered; thus an inaccuracy of one-hundredth of an inch in a division of the scale near 40 inches would make an appreciable difference in the ratio of the actual to the probable discrepancy. Also it should be noted in the present case that the observed individuals came from different parts of Scotland, so that the "original community" was really heterogeneous; and it is likely that the measurements in different regiments were taken by different observers, with different personal equations, and were not taken with as great care as would be observed at the present day. On the other hand, as the exact measurements are not given, but only the measurements to the nearest inch, the values of L_1 and of a are fitted more closely to the class-indices than they should be; and the probable discrepancy should therefore be slightly less than that given by the theoretical formula.

PART IV.—APPLICATION TO NORMAL CORRELATION.

(1.) *Correlation-Solid of Two Attributes.*

§ 25. *Correlation-Solid in General.*—Let the values of L and of M, the measures of two coexistent attributes A and B, be distributed in any manner whatever. Let L_1 and M_1 be the means, and a^2 and b^2 the mean squares of deviation from the mean. Then we know that the mean value of $(L - L_1)(M - M_1)$ is less than ab. Let this mean value be $ab \cos D$; then the angle D will be called the *divergence* of the two distributions.

Take two lines OX, OY, including an angle $\pi - D$, and on OXY as base-plane construct the solid of frequency of values of $(L - L_1)/a \sin D$ and $(M - M_1)/b \sin D$, these values being measured parallel to OX and OY respectively. Thus if we draw Ox at right angles to OY, and Oy at right angles to OX, and if on Ox and Oy respectively we take $ON' = x'$, $ON'' = x''$, and $On' = y'$, $On'' = y''$, then the portion of the solid included between planes through N′ and N″ at right angles to ON′N″ and planes through n' and n'' at right angles to $On'n''$ includes all the elements representing individuals for which L lies between $L_1 + ax'$ and $L_1 + ax''$, and M between $M_1 + by'$ and $M_1 + by''$. This solid will be called the *correlation-solid* of the two distributions. The ordinates are supposed to be measured on such a scale that the total volume of the solid is unity.

Let $L' = lL + mM$, $M' = l'L + m'M$, and let the means, mean squares of deviation, and mean product of deviation of L′ and M′ be respectively L'_1, M'_1, a'^2, b'^2, and $a'b' \cos D'$. Then

$$L' = lL_1 + mM'_1, \qquad M' = l'L_1 + m'M_1,$$

$$a'^2 = l^2a^2 + 2lmab \cos D + m^2b^2,$$

$$b'^2 = l'^2a^2 + 2l'm'ab \cos D + m'^2b^2,$$

$$a'b' \cos D' = ll'a^2 + (lm' + l'm)\, ab \cos D + mm'b^2.$$

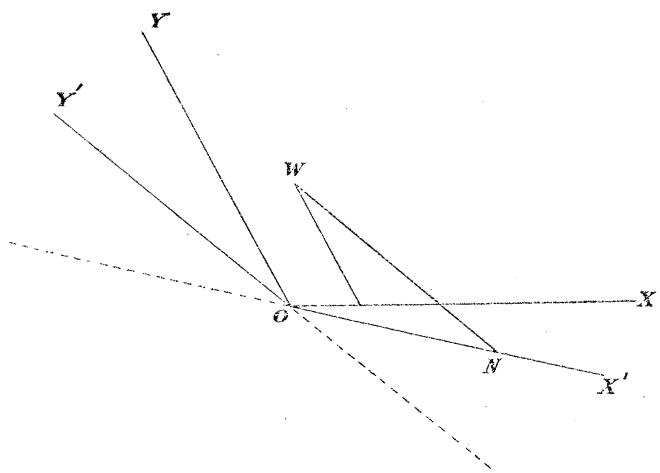
Let WR be any ordinate of the correlation-solid, the co-ordinates of W with regard to OX and OY being x cosec D and y cosec D; and let $a'x' = lax + mby$, $b'y' = l'ax + m'by$. Then WR is proportional to the number of individuals for which $L = L_1 + ax$ and $M = M_1 + by$, and therefore it is proportional to the number for which $L' = L'_1 + a'x'$, $M' = M'_1 + b'y'$. Through O draw the lines OY′, OX′, whose equations referred to OX and OY as axes are $lax + mby = 0$, $l'ax + m'by = 0$; and draw WN parallel to Y′O, meeting OX′ in N (fig. 9). Then $ON \sin X'OY' = (lax + mby)/\{l^2a^2 + 2lmab \cos D + m^2b^2\}^{\frac{1}{2}} = x'$; and similarly $NW \sin X'OY' = y'$. Hence the solid is the solid of frequency of values of $(L' - L'_1)/a' \sin X'OY'$ and $(M' - M'_1)/b' \sin X'OY'$, these values being measured parallel to OX′ and OY′ respectively. Also

$$\cos(\pi - X'OY') = \{ll'a^2 + (lm' + l'm)ab\cos D + mm'b^2\}$$
$$/ \{(l^2a^2 + 2lmab\cos D + m^2b^2)(l'^2a^2 + 2l'm'ab\cos D + m'^2b^2)\}^{\frac{1}{2}}$$
$$= \cos D',$$

and therefore $X'OY' = \pi - D'$. Hence the solid is the correlation-solid of the distributions of $lL + mM$ and $l'L + m'M$, OX′ and OY′ being taken as axes.

Thus the correlation-solid of the distributions of L and M is the same as the correlation-solid of the distributions of $lL + mM$ and $l'L + m'M$, where l, m, l', m' are any constants whatever.*

Fig. 9.



It may be noted that if D_1 and D_2 are the divergences of the distribution of $lL+mM$ from the distributions of L and of M, we have $D = D_1 + D_2$. Or, generally, if the divergence may be supposed to be either positive or negative, and if L, M, N are measures connected by a linear relation $lL + mM + nN = 0$, their divergences D, D′, D″ from one another are subject to the relation $D + D' + D'' = 0$.

§ 26. *Correlation-Solid for Normal Distributions.*—(i.) Now suppose that the distribution of L is correlated with that of M, *i.e.*, that the values of M are distributed normally with mean square b^2, and that for any particular value of M the values of L are distributed normally with constant mean square β^2 about a mean value $L_1 + \lambda(M - M_1)$, where λ is a constant. Then (§ 14) we may write $L - L_1 = \lambda(M - M_1) + L'$, where L′ is a measure whose values are distributed normally with mean square β^2 about a mean value zero, this distribution being independent of that of M. Hence the mean square of $L - L_1$ is $\lambda^2b^2 + \beta^2$, and the mean product of $L - L_1$ and $M - M_1$ is λb^2; so that, if a^2 is the mean square of $L - L_1$, we have $\lambda = a/b \,.\, \cos D$, $\beta^2 = a^2\sin^2 D$. Thus for any particular value of $(M - M_1)/b\sin D$ the values of $(L - L_1)/a\sin D$ are distributed normally with mean

* We must, of course, allow for the possibility of two solids, which really are identical, appearing to be the "reflexions" of one another.

square unity about a mean value $\{(M - M_1)/b \sin D\} \cos D$. Hence the correlation-solid is a projective solid whose vertical sections by planes parallel to OX are normal figures of semi-parameter unity; and since the values of $(M - M_1)/b$ are distributed normally with mean square unity, the sections by planes at right angles to OX are also normal figures of semi-parameter unity; *i.e.*, the correlation-solid is the standard normal solid.

(ii.) By taking vertical sections parallel to OY, we see that the values of $(L - L_1)/a$ are normally distributed, so that the values of L are normally distributed; and that in any class distinguished by a particular value of L the values of M are distributed normally with mean square $b^2 \sin^2 D$ about a mean value $M_1 + \frac{b}{a} \cos D . (L - L_1)$. In other words, if the distribution of L is correlated with that of M, the distribution of M is correlated with that of L.

(iii.) Conversely, if the correlation-solid of two distributions is the standard normal solid, the distributions are normal and normally correlated.

(iv.) We have already seen (§ 14) that when the distributions of L and of M are normally correlated, the values of $lL + mM$ are distributed normally. We might obtain this result directly by the method adopted at the beginning of § 13. In the base-plane draw the lines whose equations, referred to OX and OY as axes, are $la \sin D . x + mb \sin D . y = \xi_1$, and $la \sin D . x + mb \sin D . y = \xi_2$. Then the vertical planes through these lines will include between them the elements representing individuals for which $l(L - L_1) + m(M - M_1)$ lies between ξ_1 and ξ_2. Draw the central vertical plane at right angles to these planes, cutting the two sections in the ordinates W_1R_1 and W_2R_2. Then the number of these individuals is proportional to the area $W_1R_1R_2W_2$, *i.e.*, it is proportional to the area of the standard normal figure included between ordinates at distances $\xi_1/\{l^2a^2 + 2lmab \cos D + m^2b^2\}^{\frac{1}{2}}$ and $\xi_2/\{l^2a^2 + 2lmab \cos D + m^2b^2\}^{\frac{1}{2}}$ from the median; and therefore the values of $lL_1 + mM_1$ are distributed normally with mean square $l^2a^2 + 2lmab \cos D + m^2b^2$ about the mean value $lL_1 + mM_1$.

(v.) Since (§ 25) the correlation-solid of the distributions of $lL + mM$ and of $l'L + m'M$ is also the standard normal solid, it follows (see (iii.) above) that these two distributions are normally correlated.

§ 27. *Determination of Divergence by Double Median Classification.*—The portion of the solid which lies on the positive side of each of the two planes OZY and OZX (OZ being the axis of the solid) represents all the individuals for which L and M are greater than L_1 and M_1 respectively; and the portion which lies on the negative side of OZY and the positive side of OZX represents those for which L is less than L_1 and M greater than M_1. But, since the solid is a solid of revolution, these volumes are in the ratio of $\pi - D : D$. Hence, if we arrange the whole number of individuals in four classes, thus :—

	Below L_1.	Above L_1.
Below M_1 . .	P	R
Above M_1 . .	R	P

the divergence is equal to $\frac{R}{P+R}\pi$.*

§ 28. *Calculation of Table of Double Classification.*—In the base-plane draw Ox, Oy at right angles to OX, OY, and therefore including an angle D. In Ox take $ON = (X - L_1)/a$, $ON' = (X' - L_1)/a$; and in Oy take $On = (Y - M_1)/b$, $On' = (Y' - M_1)/b$. Through these points draw vertical planes at right angles to Ox and Oy respectively; then (§ 25) the volume of the portion of the standard solid included between these four planes represents the proportion of individuals for which L lies between X and X′ and M between Y and Y′.

The calculation of this volume requires the use of the integral calculus. For a rough calculation we may use either of two methods.

(1.) The planes by which the volume is bounded will meet the base-plane in lines forming a parallelogram, two of the sides of the parallelogram being at right angles to Ox, at distances $(X - L_1)/a$ and $(X' - L_1)/a$ from O, and the other two at right angles to Oy, at distances $(Y - M_1)/b$ and $(Y' - M_1)/b$ from O. Now suppose that the base-plane is divided up into very small areas such that the portions of the solid lying above these areas are all equal. Then the ratio of the number of these areas which lie inside the parallelogram to the total number will be the proportion of individuals for which L lies between X and X′, and M between Y and Y′. For effecting this division of the base-plane into small areas we can use either of the two characteristic properties of the normal solid.

(i.) The solid is a projective solid. Hence if we find the values of x corresponding to $\alpha = \pm 1/m$, $\alpha = \pm 2/m$, . . . $\alpha = \pm (m-1)/m$, and if we take the corresponding points on each of two rectangular axes $\xi'O\xi$, $\eta'O\eta$ in the base-plane, and draw lines through these points parallel to $\eta'O\eta$ and to $\xi'O\xi$ respectively, the two sets of lines will divide the base into $4m^2$ areas, corresponding to the division of the solid into $4m^2$ equal portions. Fig. 10 shows the arrangement of these lines for $m = 50$; thus the figure contains 10,000 rectangles (one or two of the sides of some of them being at infinity), and each rectangle represents 1/10,000 of the whole volume of the solid. The centre O of the figure is shown by a small circle. The larger circle is introduced to show the scale; its radius is the semi-parameter of the solid, and is therefore the unit for measuring the distances $(X - L)/a$, &c.

The values of x corresponding to $m = 100$ are given in Table VI. (p. 167); so that

* This formula obviously applies in any case in which the correlation-solid is a solid of revolution.

by means of this table we can divide the base into 40,000 areas, each representing 1/40,000 of the whole volume. To simplify the counting of the areas, every tenth line should be drawn in ink, the others being in pencil; a dot should be placed in each area, and the pencil lines should then be erased. There will thus be 400 larger areas, each containing 100 dots. It will be found convenient to replace the circle shown in fig. 10 by a larger graduated circle; if the radius of this circle is ρ, and if Ox cuts the circumference of the circle in F, the line at right angles to Ox at a distance x from O will cut the circumference in points at an angular distance $\cos^{-1} x/\rho$ from F.

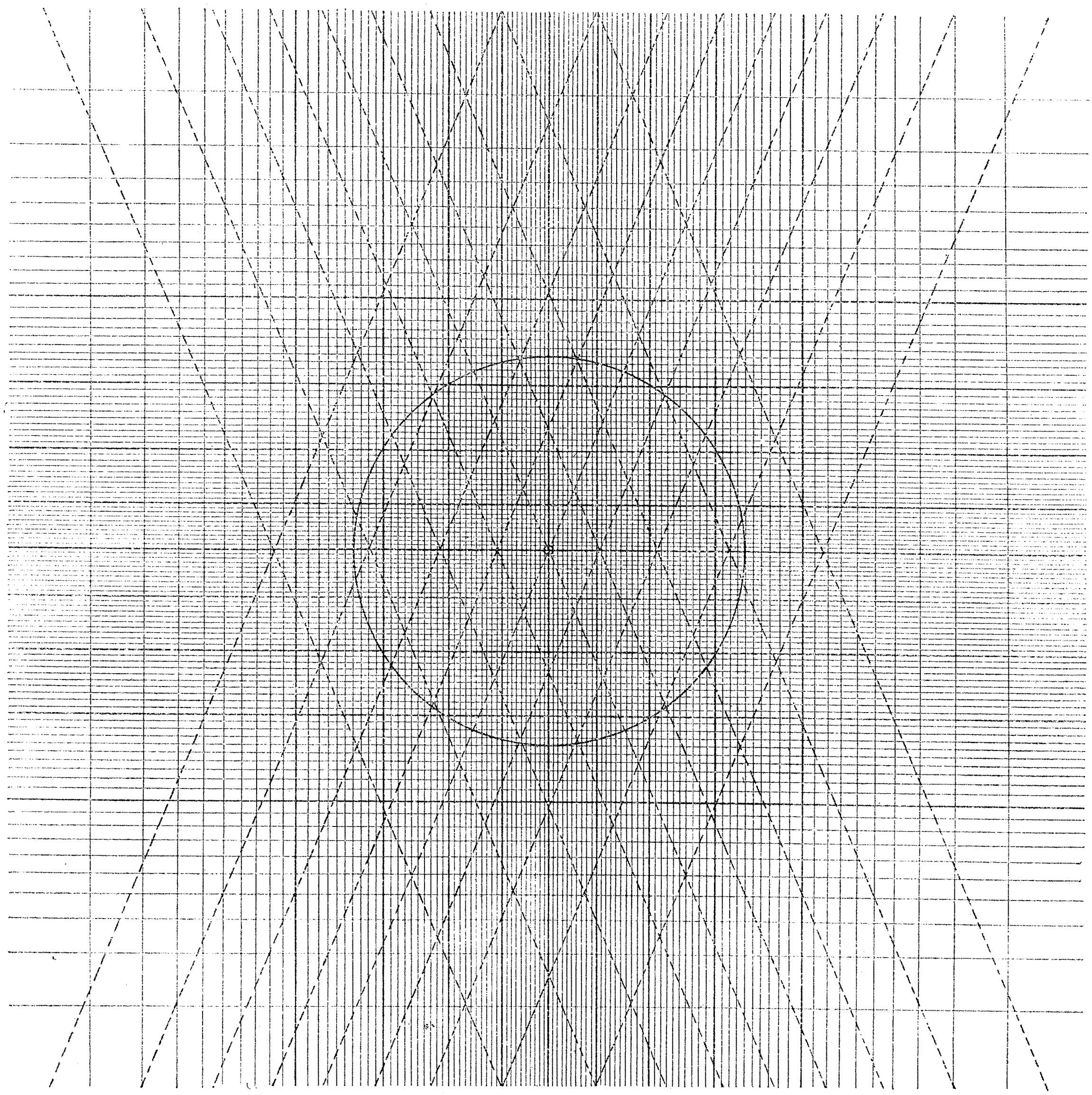
The lines Ox, Oy, &c., may be shown on tracing-paper, instead of on the figure itself; and the paper may then be turned round O into two or three different positions, so as to minimise inaccuracies of counting. Or the figure may be copied on to a glass plate, and the lines Ox, Oy, &c., drawn on ordinary paper.

(ii.) The solid is a solid of revolution, and therefore can be divided into mm' equal portions by a set of m planes through the central ordinate at successive angular distances $2\pi/m$, and a set of concentric cylinders enclosing portions $1/m'$, $2/m'$, ... $(m'-1)/m'$ of the whole volume. Let the rth cylinder cut a central section in the ordinate MP. Then, if OH is the central ordinate, $r/m' = (\text{OH} - \text{MP})/\text{OH}$ (§§ 5, 11). Hence the radii of the successive cylinders are the abscissæ of the standard curve corresponding to ordinates whose ratios to the central ordinate are respectively $(m'-1)/m'$, $(m'-2)/m'$, ... $1/m'$. Thus for $m' = 100$ the values are given by Table II. (p. 155).

This method of division of the base-plane is not so convenient as the method explained in (i.), but it may be used for testing the accuracy of a figure constructed according to that method. If on such a figure we draw circles with the radii given by Table II., each of the rings so formed should contain one-hundredth of the total number of dots in the figure. Or, if we draw circles with radii ·05, ·10, ·15, ..., the numbers in the successive rings should be proportional to the differences shown in the fourth column of Table I.

(2.) A more accurate method can be adopted when the values of X and X′, and also those of Y and Y′, have been chosen so as to correspond to particular class-indices. Let these be α, α', β, and β' respectively, and let the corresponding abscissæ of the standard normal figure be x, x', y, and y'. Thus $(\text{X} - \text{L}_1)/a = x$, $(\text{X}' - \text{L}_1)/a = x'$, $(\text{Y} - \text{M}_1)/b = y$, $(\text{Y}' - \text{M}_1)/b = y'$. Now if, by the method of § 11, we construct a figure representing the division of the standard solid by parallel vertical planes at distances x and x' from OH, and also a corresponding figure for distances y and y', the bases of the two figures being in the same straight line, and the distance between corresponding extremities being equal to $\text{D}/2\pi$ of either base, the area formed by the two pairs of curves will give the proportion of individuals for which L lies between X and X′, and Y between Y and Y′. The most important case is that in which the class-indices for each distribution separately correspond to the division of the community into p numerically equal classes. The table of double classification of values of L and of M will then contain p^2 compartments; and if we draw the figure corresponding to the division of the standard solid into p equal portions by parallel vertical planes, and shift this figure along its base through a

Fig. 10.



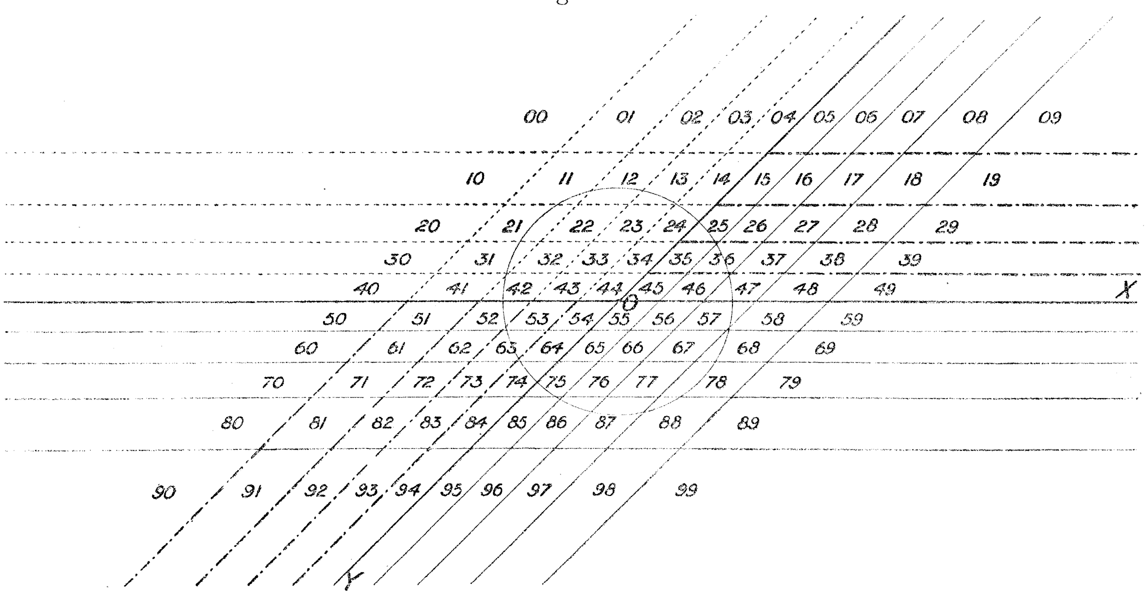
distance equal to $D/2\pi$ of its whole breadth (the part of the figure which projects on one side being superposed on the other side, so as to leave the whole breadth unaltered), we obtain a diagram with p^2 compartments, whose areas are proportional to the numbers in the corresponding compartments of the table of double classification.

Suppose, for instance, that $p = 10$. If $X_1, X_2, \ldots X_9$ and $Y_1, Y_2, \ldots Y_9$ denote the "decile" values of L and of M respectively, the table of double classification will be of this form:—

Values of M.	Values of L.									
	$-\infty$ to L_1.	L_1 to L_2.	L_2 to L_3.	L_3 to L_4.	L_4 to L_5.	L_5 to L_6.	L_6 to L_7.	L_7 to L_8.	L_8 to L_9.	L_9 to $+\infty$.
$-\infty$ to M_1	(00)	(01)	(02)	(03)	(04)	(05)	(06)	(07)	(08)	(09)
M_1 to M_2	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)
M_2 to M_3	(20)	(21)	(22)	(23)	(24)	(25)	(26)	(27)	(28)	(29)
M_3 to M_4	(30)	(31)	(32)	(33)	(34)	(35)	(36)	(37)	(38)	(39)
M_4 to M_5	(40)	(41)	(42)	(43)	(44)	(45)	(46)	(47)	(48)	(49)
M_5 to M_6	(50)	(51)	(52)	(53)	(54)	(55)	(56)	(57)	(58)	(59)
M_6 to M_7	(60)	(61)	(62)	(63)	(64)	(65)	(66)	(67)	(68)	(69)
M_7 to M_8	(70)	(71)	(72)	(73)	(74)	(75)	(76)	(77)	(78)	(79)
M_8 to M_9	(80)	(81)	(82)	(83)	(84)	(85)	(86)	(87)	(88)	(89)
M_9 to $+\infty$	(90)	(91)	(92)	(93)	(94)	(95)	(96)	(97)	(98)	(99)

The corresponding portions of the standard solid will be bounded by planes whose intersections with the base-plane will form a "plan" such as the following (fig. 11*):—

Fig. 11.

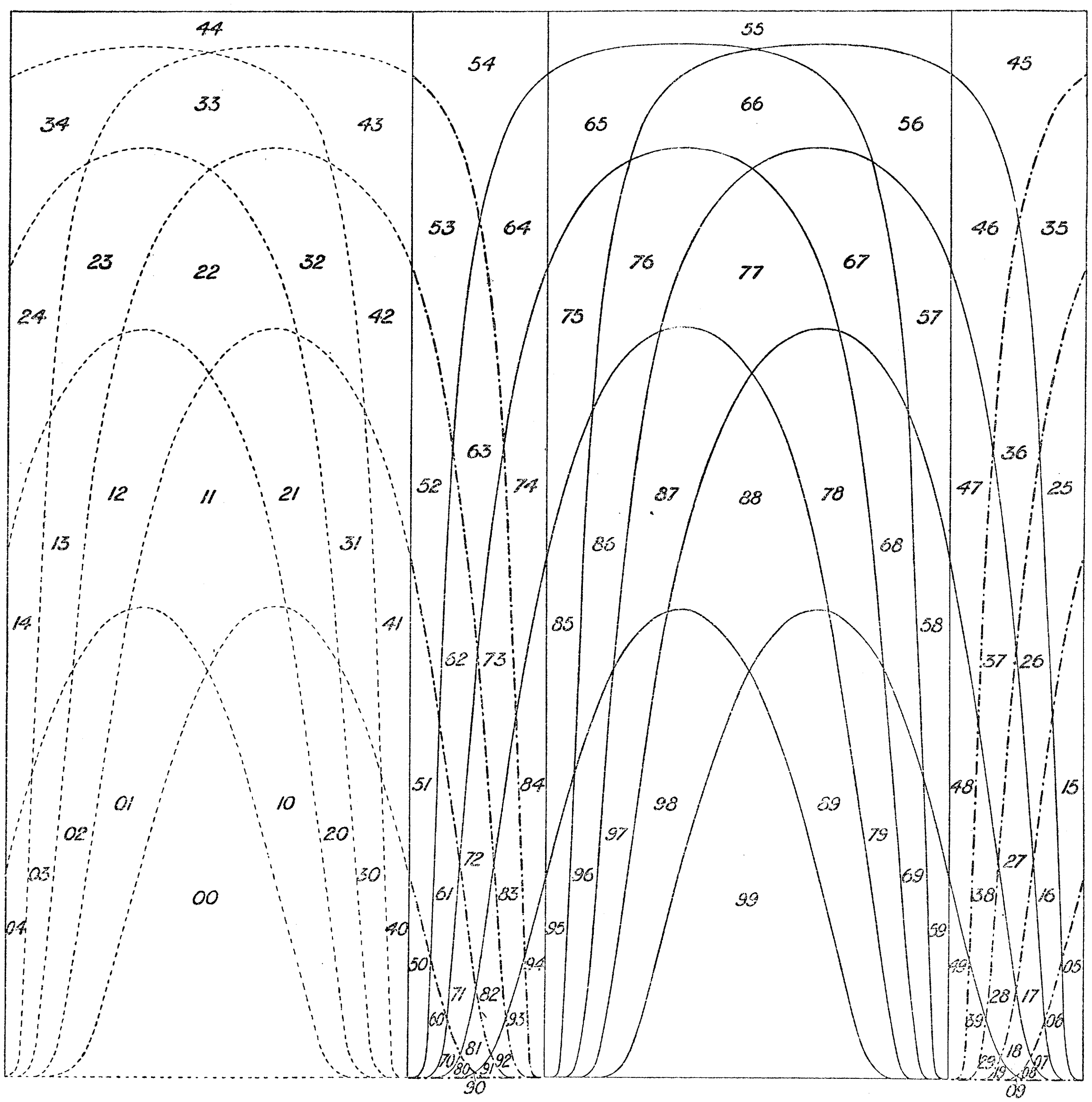


and the volumes of these portions are equal to the one hundred compartments in the diagram formed by shifting fig. 7 (omitting the alternate curves, which correspond to the values ·1, ·3, ·5, ·7, and ·9 of α) through the required distance.

* In this figure, as in fig. 10, the base-plane is supposed to be seen from above. In fig. 9 it is seen from below.

Fig. 12 shows the form of this diagram for the case of $D = \frac{1}{4}\pi$, so that it is produced by shifting fig. 7 from right to left through one-eighth of its whole breadth. The dotted lines in fig. 10 (p. 143) show the position of the corresponding planes dividing the standard solid into 100 portions; the angle between the two sets of lines

Fig. 12.



is $\frac{1}{4}\pi$ (or $\frac{3}{4}\pi$), and the distances of the lines in either set from the foot of the central ordinate are respectively ·25335, ·52440, ·84162, and 1·28155, the radius of the circle shown in the figure being the unit. If instead of dividing up the base-plane, as in fig. 10, in the manner explained in (1.) (i.) above, we had divided it up by 100 radial lines and 99 circles as explained in (1.) (ii.), these lines and circles would become vertical and horizontal straight lines dividing the diagram (fig. 12) into 10,000 equal squares.

For practical applications of this method, it is sufficient to have the single figure as shown in fig. 7. The curves representing the displacement of the figure through the distance $D/2\pi$ can then be traced by means of a double-barred parallel rule or an antigraph. But it is better to draw the curves directly from Tables III. and IV.*

§ 29. *Differential Relation of* V *and* D.—Let V denote the proportion of individuals for which L exceeds X, and M exceeds Y. Then V is the volume of the solid lying on the positive side of the vertical planes drawn through N and n (§ 28) at right angles to ON and On respectively. Let the sections of the solid by these planes intersect in the ordinate WR; and let them meet the base-plane in the straight lines NWη and nWξ respectively.

Let $V - v$ denote the value which V would have if the divergence, instead of being D, were $D + \theta$, the values of ON and On being unaltered. This alteration in the value of V might be obtained by keeping ON and Nη fixed, and rotating On and $n\xi$ about OZ through an angle θ. Now suppose that θ is very small. Then the consecutive positions of the vertical section through $n\xi$ will intersect close to the ordinate at n; and therefore v is the volume obtained by rotating the area WRξ about the ordinate at n through an angle θ. Hence, for a first approximation, we have $v = \text{WR}.\theta$ (§ 5).

We might have obtained this result by considering the alteration, due to the change of D into $D + \theta$, of the diagram constructed in the manner explained in the last section. The area which is equal to V is bounded by the base and by two curves intersecting at a point whose height above the base is $2\pi.\text{WR}$; and the decrement v is obtained by shifting one curve laterally through a distance $\theta/2\pi$. Hence $v = \text{WR}.\theta$. Let the two curves, at their point of intersection, be inclined to the base at angles ω_1 and ω_2. Then it will be seen that for a second approximation we have $v = \text{WR}.\theta + \frac{1}{2}\sin\omega_1\sin\omega_2\operatorname{cosec}(\omega_1 + \omega_2).(\theta/2\pi)^2$.

The ordinate WR is the ordinate, for abscissa $(x^2 - 2xy\cos D + y^2)^{\frac{1}{2}}\operatorname{cosec} D$, of

* It has been suggested that the one set of curves might be drawn on a board or stiff card, and the other on a thin sheet of some transparent substance (*e.g.*, of talc), which could be slipped across the face of the card. This, however, might require the curves to be drawn on too small a scale to be really useful.

Table III. can be used for drawing the curve corresponding to any value of α not given in the table. If x and x' are the abscissæ of the standard curve corresponding to class-indices α and α', the equations to the corresponding curves of the divergence-diagram are $z = \exp(-\frac{1}{2}x^2\sec^2 2\pi\theta)$ and $z' = \exp(-\frac{1}{2}x'^2\sec^2 2\pi\theta)$. Hence, for any particular value of θ, we have $\log z'/\log z = x'^2/x^2$. The value of z being given by the table, the value of z' may be deduced by means of an ordinary slide-rule and a pair of proportional compasses.

The methods described in the text can be extended to the problems which occur in the theory of the error in the position of a point in a plane (as in BRAVAIS' memoir, referred to by Professor PEARSON). Thus the condition that the point lies within an area limited by the curve $f(x, y) = 0$ is found by taking the curve Σ of § 11 to be the curve whose equation, referred to axes including an angle $\pi - D$, is $f(x/a \sin D,\ y/b \sin D) = 0$, and then counting the dots or measuring the corresponding cylinder-area.

the normal curve of semi-parameter unity and central ordinate $1/2\pi$ (area $= 1/\sqrt{2\pi}$); where $x = (X - L_1)/a$, $y = (Y - M_1)/b$.

Applications of the Theory of Error.

§ 30. *Probable Error in Value of Divergence, as Obtained by Different Methods.*—Let the distributions of L and of M be normally correlated, the means, mean squares of deviation, and mean product of deviation, being L_1, M_1, a^2, b^2, and $ab \cos D$. If a random selection of n individuals is made, the divergence can be found by any one of several different methods. We require to find the probable error in D, due to the use of each method.

(1.) Suppose that we take the averages, average squares, and average product, as equal to the means, mean squares, and mean product for the complete community. The general expression for the resulting probable error in $\cos D \equiv S_{1,1}/\sqrt{\lambda_2\mu_2}$ has been found in § 20. To find the values of $S_{3,1}$, $S_{2,2}$, and $S_{1,3}$, in the case of normal correlation, we write $L - L_1 = (a/b) \cos D . (M - M_1) + L'$; then $M - M_1$ and L' are independent, and their mean squares are respectively b^2 and $a^2 \sin^2 D$. The mean fourth power of $M - M_1$ is $3b^4$; and thus we find $S_{3,1} = 3a^3b \cos D$, $S_{2,2} = a^2b^2(1 + 2\cos^2 D)$, $S_{1,3} = 3ab^3 \cos D$. The table in § 20 becomes

	λ_2	$S_{1,1}$	μ_2
λ_2	$2a^4$	$2a^3b \cos D$	$2a^2b^2 \cos^2 D$
$S_{1,1}$		$a^2b^2(1 + \cos^2 D)$	$2ab^3 \cos D$
μ_2			$2b^4$

and hence we find that the probable error in D, due to adopting this method, is $Q \sin D/\sqrt{n}$.

(2.) Let D be determined by the method of § 27. Let the medians as given by the data be respectively L'_1 and M'_1, and let the result of the double median classification be

	Below L'_1.	Above L'_1.
Below M'_1	P′	R′
Above M'_1	R′	P′

so that $n = 2(P' + R')$. Let $P = n(\pi - D)/2\pi$, $R = nD/2\pi$; and let the classification of the observed individuals with regard to the true means of the complete community be

	Below L_1.	Above L_1.
Below M_1	(D) $P - \theta - \phi - \psi$	(C) $R + \psi$
Above M_1	(B) $R + \phi$	(A) $P + \theta$

Then the erroneous values L'_1 and M'_1 are obtained by shifting the medians so that this table may present the appearance of the former table. Thus L_1 is shifted so as to transfer $\theta + \psi$ individuals, and M_1 is shifted so as to transfer $\theta + \phi$. In the first case the particular individuals are in the class for which $L = L_1$ (to a first approximation); and the median of M for this class is at M_1, so that half of the $\theta + \psi$ are put from class (C) into class (D), and half from class (A) into class (B). Similarly half of the $\theta + \phi$ are put from (A) into (C), and half from (B) into (D). Hence

$$P' = P - \tfrac{1}{2}\phi - \tfrac{1}{2}\psi, \qquad R' = R + \tfrac{1}{2}\phi + \tfrac{1}{2}\psi;$$

and the error in D is

$$\tfrac{1}{2}\pi(\phi + \psi) / (P + R) = \pi(\phi + \psi) / n.$$

This error is distributed with mean square $D(\pi - D) / n$; and therefore the probable error in D as obtained by the second method is $Q\sqrt{D(\pi - D)} / \sqrt{n}$.

This probable error is of course greater than the probable error due to using the method of (1.), since $\sqrt{D(\pi - D)} > \sin D$.

(3.) Suppose that, instead of taking the medians, we fix on any two class-indices α and β, and divide the total community into four classes (A), (B), (C), and (D) by a double classification with regard to the corresponding values X and Y of L and M respectively, thus:—

	Below X.	Above X.	Total.
Below Y	(D) $\tfrac{1}{2}(\alpha + \beta) + V = V'''$	(C) $\tfrac{1}{2}(1 - \alpha) - V = V''$	$\tfrac{1}{2}(1 + \beta)$
Above Y	(B) $\tfrac{1}{2}(1 - \beta) - V = V'$	(A) V	$\tfrac{1}{2}(1 - \beta)$
Total	$\tfrac{1}{2}(1 + \alpha)$	$\tfrac{1}{2}(1 - \alpha)$	1

The value of V is different for different values of D. But, if x and y are the abscissæ of the standard normal figure corresponding to the class-indices α and β, it is easily seen that V depends solely on x, y, and D. Hence, if we choose α and β, and observe V, D is (theoretically) determined.

Let the errors in the values of X and of Y be ξ and η. Then the observed value of V is the proportion of individuals for which L exceeds $X + \xi$ and M exceeds $Y + \eta$. Let the actual nubmers coming from the four classes (A), (B), (C), and (D) be $n(V + \psi)$, $n(V' + \psi')$, $n(V'' + \psi'')$, and $n(V''' + \psi''')$; thus $\psi + \psi' + \psi'' + \psi''' = 0$. Let the areas of the sections of the standard solid by the planes NWRη and nWRξ (§ 29) be Γ and Δ, and let these areas be divided by WR in the ratios of $1 + \gamma : 1 - \gamma$ and $1 + \delta : 1 - \delta$ respectively. Thus Γ and Δ are equal to the ordinates of the standard figure corresponding to abscissæ x and y (class-indices α and β); while γ is the class-index of Y in the class for which L = X, and δ is the class-index of X in the class for which M = Y, these being the class-indices corresponding to abscissæ $(y - x \cos D)/\sin D$ and $(x - y \cos D)/\sin D$ in the standard figure.

The erroneous values $X + \xi$ and $Y + \eta$ are obtained by transferring $n(\psi + \psi'')$ individuals from (A) and (C) to (B) and (D), and $n(\psi + \psi')$ from (A) and (B) to (C) and (D). The first transfer takes place (to our order of approximation) in the class for which L = X, and the second in the class for which M = Y; so that the proportion appearing to fall in (A) is

$$\begin{aligned} V + \psi - \tfrac{1}{2}(1-\gamma)(\psi + \psi'') - \tfrac{1}{2}(1-\delta)(\psi + \psi') \\ = V + \tfrac{1}{2}(1+\gamma)\,.\,\tfrac{1}{2}(1+\delta)\,.\,\psi - \tfrac{1}{2}(1+\gamma)\,.\,\tfrac{1}{2}(1-\delta)\,.\,\psi' \\ - \tfrac{1}{2}(1-\gamma)\,.\,\tfrac{1}{2}(1+\delta)\,.\,\psi'' + \tfrac{1}{2}(1-\gamma)\,.\,\tfrac{1}{2}(1-\delta)\,\psi''' \\ = V + \Psi. \end{aligned}$$

Let WR = Z. Then the error Ψ in V produces (§ 29) an error $-\Psi/Z$ in D, and therefore the probable error in D, as determined by this method, is

$$Q\,.\,\Theta/\sqrt{n},$$

where

$$\begin{aligned} \Theta^2 = \tfrac{1}{16}Z^{-2}[\{V(\overline{1+\gamma}\,.\,\overline{1+\delta})^2 + V'(\overline{1+\gamma}\,.\,\overline{1-\delta})^2 + V''(\overline{1-\gamma}\,.\,\overline{1+\delta})^2 + V'''(\overline{1-\gamma}\,.\,\overline{1-\delta})^2\} \\ - \{V\,.\,\overline{1+\gamma}\,.\,\overline{1+\delta} - V'\,.\,\overline{1+\gamma}\,.\,\overline{1-\delta} - V''\,.\,\overline{1-\gamma}\,.\,\overline{1+\delta} + V'''\,.\,\overline{1-\gamma}\,.\,\overline{1-\delta}\}^2]. \end{aligned}$$

Since $V + V' + V'' + V''' = 1$, $V + V' = \tfrac{1}{2}(1 - \beta)$, $V + V'' = \tfrac{1}{2}(1 - \alpha)$, this probable error can be expressed in terms of V, α, β, γ, δ. But the above is the most symmetrical form, and the most convenient for calculation.

(4.) By taking a number of different values of α and β, and observing the corresponding values of V, we get a series of values of D; and then we can take the weighted mean of these, the weights being assigned in such a way as to make the probable error as small as possible.

§ 31. *Relative Accuracy of the Different Methods.*—By means of § 5 it may be shown that, with the notation of § 20 and § 30 (3.),

$$\sigma_{1,0} = a\,\{\tfrac{1}{2}\Gamma(1-\gamma) + \tfrac{1}{2}\Delta(1-\delta)\cos D\},$$
$$\sigma_{0,1} = b\,\{\tfrac{1}{2}\Gamma(1-\gamma)\cos D + \tfrac{1}{2}\Delta(1-\delta)\},$$
$$\sigma_{2,0} = a^2\{V + Z\sin D\cos D + \tfrac{1}{2}\Gamma(1-\gamma)x + \tfrac{1}{2}\Delta(1-\delta)y\cos^2 D\},$$
$$\sigma_{1,1} = ab\{V\cos D + Z\sin D + \tfrac{1}{2}\Gamma(1-\gamma)x\cos D + \tfrac{1}{2}\Delta(1-\delta)y\cos D\},$$
$$\sigma_{0,2} = b^2\{V + Z\sin D\cos D + \tfrac{1}{2}\Gamma(1-\gamma)x\cos^2 D + \tfrac{1}{2}\Delta(1-\delta)y\}.$$

Thus from § 20 we have the following table :—

	L_1	M_1	a	b	D	V
L_1	a^2	$ab\cos D$	0	0	0	$a\,\{\frac{1}{2}\Gamma(1-\gamma) + \frac{1}{2}\Delta(1-\delta)\cos D\}$
M_1		b^2	0	0	0	$b\,\{\frac{1}{2}\Gamma(1-\gamma)\cos D + \frac{1}{2}\Delta(1-\delta)\}$
a			$\frac{1}{2}a^2$	$\frac{1}{2}ab\cos^2 D$	$-\frac{1}{2}a\sin D\cos D$	$\frac{1}{2}a\,\{Z\sin D\cos D + \frac{1}{2}\Gamma(1-\gamma)x + \frac{1}{2}\Delta(1-\delta)y\cos^2 D\}$
b				$\frac{1}{2}b^2$	$-\frac{1}{2}b\sin D\cos D$	$\frac{1}{2}b\,\{Z\sin D\cos D + \frac{1}{2}\Gamma(1-\gamma)x\cos^2 D + \frac{1}{2}\Delta(1-\delta)y\}$
D					$\sin^2 D$	$-[Z\sin^2 D + \frac{1}{2}\{\frac{1}{2}\Gamma(1-\gamma)x + \frac{1}{2}\Delta(1-\delta)y\}\sin D\cos D]$
V						$V(1-V)$

Let the errors in L_1, M_1, a, b, D, be ω, ω', ρ, ρ', θ. The error in V is ψ; if we write this $= \frac{1}{2}\Gamma(1-\gamma)(\omega + x\rho)/a + \frac{1}{2}\Delta(1-\delta)(\omega' + y\rho')/b - Z\theta + \phi$, then it may be shown by the above table that the mean products of ϕ with ω, ω', ρ, ρ', and θ are zero. By writing in the one case $\Delta = 0$, $\gamma = -1$, $Z = 0$, and in the other $\Gamma = 0$, $\delta = -1$, $Z = 0$, we see that $\psi + \psi''$ and $\psi + \psi'$ are of the forms $\Gamma(\omega + x\rho)/a + \chi$ and $\Delta(\omega' + y\rho')/b + \chi'$, where the mean products of χ or χ' with ω, ω', ρ, ρ', and θ, are zero (*cf.* § 23). Hence we obtain the following results :—

(1.) Suppose that we fix on definite values X and Y of L and M, and that we require the proportion of individuals for which L exceeds X, and M exceeds Y. If we determine L_1, M_1, a, b, and D from the averages, average squares, and average product, and then calculate the value of V, the resulting error is $\frac{1}{2}\Gamma(1-\gamma)(\omega + x\rho)/a + \frac{1}{2}\Delta(1-\delta)(\omega' + y\rho')/b - Z\theta$. The mean square of this error is less than the mean square of ψ, the error in V as obtained by direct observation;* and therefore we obtain a better result for V by the calculation than by observation.

(2.) Suppose that we fix on particular class-indices α and β, and that we require the corresponding value of V. The error in V, as determined by calculating the averages, average squares, and average product, is $-Z\theta$; while the error for direct observation is (§ 30 (3.)) $\psi - \frac{1}{2}(1-\gamma)\{\Gamma(\omega + x\rho)/a + \chi\} - \frac{1}{2}(1-\delta)\{\Delta(\omega' + y\rho')/b + \chi'\} = -Z\theta + \phi - \frac{1}{2}(1-\gamma)\chi - \frac{1}{2}(1-\delta)\chi'$. Since the mean products of ϕ, χ, χ' with θ are zero, the mean square of this last error is greater than the mean square of $-Z\theta$.

This result, of course, is identical with (1.); for if the observed class-index of X′ is α', we may consider that we are observing either the class-index of X′ or the value of L corresponding to class-index α'.

(3.) If we determine D by the method of § 30 (3.), the resulting error is $\theta - Z^{-1}\{\phi - \frac{1}{2}(1-\gamma)\chi - \frac{1}{2}(1-\delta)\chi'\}$. The mean product of θ and $\phi - \frac{1}{2}(1-\gamma)\chi - \frac{1}{2}(1-\delta)\chi'$ is zero; hence the probable error due to the method of § 30 (3.) is greater than that due to the method of § 30 (1.).

(4.) Similarly, if we take the weighted mean of a number of different values of D, as in § 30 (4.), we shall still get an error of the form $\theta + \Phi$, where the mean value of $\theta\Phi$ is zero. Hence, if the averages, average squares, and average product can be determined, the value of D so obtained cannot be improved by direct observation of the values of V corresponding to selected pairs of class-indices.†

(5.) Generally, let R be any quantity which would be known if the true means, mean squares, and mean product of L and M were known. Let R_1 be the value obtained by taking these as equal to the averages, average squares, and average product, for the n individuals; and let R_2 be the value obtained by any other method involving observation of the numbers occurring in any set of classes determined by a finite number of class-indices of L and M, with or without the use of the averages, average squares, and average products. Let Θ_1^2/n and Θ_2^2/n be the mean squares of the errors in R as determined by the two methods. Then the propositions stated in § 23 (4.) hold good. The theorem may be extended to the case of any number of mutually correlated attributes.

§ 32. *Test of Hypothesis as to Normal Correlation.*—To test whether the distributions of L and of M, in any particular case, may be regarded as normally correlated. we use the method of § 24, with the necessary modifications.

(1.) With the notation of § 31 (5.), let R denote the proportion of individuals for which L exceeds X and M exceeds Y, the values of X and Y being fixed beforehand. Then, writing $\frac{1}{2}\Gamma(1-\gamma) = A$, $\frac{1}{2}\Delta(1-\delta) = B$, we have

* This shows that $V(1-V)$ is greater than $A^2(1+\frac{1}{2}x^2) + 2AB(1+\frac{1}{2}xy\cos D)\cos D + B^2(1+\frac{1}{2}y^2) + (Ax+By)Z\sin D\cos D + Z^2\sin^2 D$, where $A = \frac{1}{2}\Gamma(1-\gamma)$, $B = \frac{1}{2}\Delta(1-\delta)$.

† *Cf.* KARL PEARSON, in 'Phil. Trans.,' A, vol. 187 (1896), p. 265.

$$\Theta_1^2 = A^2 (1 + \tfrac{1}{2} x^2) + 2AB (1 + \tfrac{1}{2} xy \cos D) \cos D$$
$$+ B^2 (1 + \tfrac{1}{2} y^2) + (Ax + By) Z \sin D \cos D + Z^2 \sin^2 D,$$

and

$$\Theta_2^2 = V (1 - V).$$

Thus the "discrepancy" is the difference between V as calculated by finding the means, mean squares, and mean product, and V as found by direct observation; and the probable discrepancy is $Q (\Theta_2^2 - \Theta_1^2)^{\frac{1}{2}}/\sqrt{n}$.

In adopting this method we are testing both the normal distribution of each measure separately and also the normal correlation of the two distributions; and therefore it is not necessary to test first whether the separate distributions are normal.

(2.) Suppose that we are satisfied that the separate distributions are normal, and that we require to test whether, on this assumption, they may be regarded as normally correlated. Then R, in § 31 (5.), will denote the proportion for which L exceeds the value found to correspond to class-index α, and M exceeds the value found to correspond to class-index β. The discrepancy is (§ 30 (3.)) the difference between the errors $- Z\theta$ and $\psi - \frac{1}{2}(1 - \gamma)(\psi + \psi'') - \frac{1}{2}(1 - \delta)(\psi + \psi')$. (This difference, by § 31 (2.), may be written in the form $\phi - \frac{1}{2}(1 - \gamma)\chi - \frac{1}{2}(1 - \delta)\chi'$.) The mean square of the discrepancy is $Z^2 (\Theta^2 - \sin^2 D)/n$, where Θ^2 has the value given in § 30 (3.); so that the probable discrepancy is $Q.Z(\Theta^2 - \sin^2 D)^{\frac{1}{2}}/\sqrt{n}$. When this method is adopted, the sum of all the discrepancies in any row or in any column of the table of double classification is zero.

(3.) In some cases we are not able either to calculate the averages, average squares, and average product, or to test whether the separate distributions are normal. We must then determine D by some other method, and proceed as in (2.). Suppose, for instance, that D is determined by the double-median-classification method of § 27. Then, as in (2.), the discrepancy is the difference between the value of V, calculated for particular class-indices α and β, and the observed value of V for these class-indices; and the probable discrepancy is $Q.\Phi/\sqrt{n}$, where Φ^2 has different forms according as α and β are positive or negative. If α and β are both positive, it may be shown that

$$\Phi^2 = D(\pi - D) Z^2 - 2(\pi - D) Z \{\tfrac{1}{2}(1 - \alpha) . \tfrac{1}{2}(1 - \gamma) + \tfrac{1}{2}(1 - \beta) . \tfrac{1}{2}(1 - \delta)\}$$
$$- 2 DZV + 2\pi Z \{\tfrac{1}{2}(1 - \gamma) W + \tfrac{1}{2}(1 - \delta) W'\} + \Theta^2 Z^2;$$

Θ^2 having the value given in § 30 (3.), and W and W′ denoting what V would become if we put $\beta = 0$ and $\alpha = 0$ respectively, without altering the value of D.

TABLES.

Table I.—Ordinate of Standard Normal Curve in terms of Abscissa.

Abscissa $= x$. Ordinate $= z$. $z = Ce^{-\frac{1}{2}x^2}$, where $C = \frac{1}{\sqrt{2\pi}}$.

x.	z.	z/C.	Differences of z/C.	x.	z.	z/C.	Differences of z/C.
·00	·39894	1·00000	125	1·25	·18265	·45783	2827
·05	·39844	·99875	374	1·30	·17137	·42956	2754
·10	·39695	·99501	620	1·35	·16038	·40202	2671
·15	·39448	·98881	861	1·40	·14973	·37531	2581
·20	·39104	·98020	1097	1·45	·13943	·34950	2485
·25	·38667	·96923	1323	1·50	·12952	·32465	2383
·30	·38139	·95600	1541	1·55	·12001	·30082	2278
·35	·37524	·94059	1747	1·60	·11092	·27804	2170
·40	·36827	·92312	1941	1·65	·10226	·25634	2059
·45	·36053	·90371	2121	1·70	·09405	·23575	1948
·50	·35207	·88250	2287	1·75	·08628	·21627	1837
·55	·34294	·85963	2436	1·80	·07895	·19790	1726
·60	·33322	·83527	2570	1·85	·07206	·18064	1617
·65	·32297	·80957	2687	1·90	·06562	·16447	1509
·70	·31225	·78270	2786	1·95	·05959	·14938	1404
·75	·30114	·75484	2869	2·00	·05399	·13534	1304
·80	·28969	·72615	2935	2·05	·04879	·12230	1205
·85	·27798	·69680	2982	2·10	·04398	·11025	1111
·90	·26609	·66698	3015	2·15	·03955	·09914	1022
·95	·25406	·63683	3030	2·20	·03547	·08892	936
1·00	·24197	·60653	3030	2·25	·03174	·07956	855
1·05	·22988	·57623	3016	2·30	·02833	·07101	780
1·10	·21785	·54607	2986	2·35	·02522	·06321	708
1·15	·20594	·51621	2946	2·40	·02239	·05613	641
1·20	·19419	·48675	2892	2·45	·01984	·04972	578

TABLE I.—Ordinate of Standard Normal Curve in terms of Abscissa (continued).

x.	z.	z/C.	Differences of z/C.
2·50	·01753	·04394	
			521
2·55	·01545	·03873	
			468
2·60	·01358	·03405	
			419
2·65	·01191	·02986	
			374
2·70	·01042	·02612	
			333
2·75	·00909	02279	
			295
2·80	·00792	·01984	
			261
2·85	·00687	·01723	
			231
2·90	·00595	·01492	
			203
2·95	·00514	·01289	
			178
3·00	·00443	·01111	
			156
3·05	·00381	·00955	
			136
3·10	·00327	·00819	
			119
3·15	·00279	·00700	
			102
3·20	·00238	·00598	
			89
3·25	·00203	·00509	
			77
3·30	·00172	·00432	
			66
3·35	·00146	·00366	
			57
3·40	·00123	·00309	
			49
3·45	·00104	·00260	
			41
3·50	·00087	·00219	
			36
3·55	·00073	·00183	
			30

x.	z.	z/C.	Differences of z/C.
3·60	·00061	·00153	
			25
3·65	·00051	·00128	
			22
3·70	·00042	·00106	
			18
3·75	·00035	·00088	
			15
3·80	·00029	·00073	
			13
3·85	·00024	·00060	
			10
3·90	·00020	·00050	
			9
3·95	·00016	·00041	
			7
4·00	·00013	·00034	
			12
4·10	·00009	·00022	
			7
4·20	·00006	·00015	
			5
4·30	·00004	·00010	
			4
4·40	·00002	·00006	
			2
4·50	·00002	·00004	
			1
4·60	·00001	·00003	
			1
4·70	·00001	·00002	
			1
4·80	·00000	·00001	
			0
4·90	·00000	·00001	
			1
5·00	·00000	·00000	
			100000

TABLE II.—Abscissa of Standard Normal Curve in terms of Ordinate.

(*Converse of Table I.*)

z/C.	x.	z/C.	x.	z/C.	x.
1·00	·0000000	·66	·9116090	·33	1·4890686
·99	·1417768	·65	·9282057	·32	1·5095922
·98	·2010110	·64	·9447615	·31	1·5304790
·97	·2468166	·63	·9612861	·30	1·5517557
·96	·2857341	·62	·9777891	·29	1·5734512
·95	·3202914	·61	·9942800	·28	1·5955975
·94	·3517823	·60	1·0107677	·27	1·6182295
·93	·3809743	·59	1·0272612	·26	1·6413858
·92	·4083665	·58	1·0437693	·25	1·6651092
·91	·4343056	·57	1·0603008	·24	1·6894475
·90	·4590436	·56	1·0768644	·23	1·7144538
·89	·4827708	·55	1·0934688	·22	1·7401883
·88	·5056350	·54	1·1101226	·21	1·7667189
·87	·5277539	·53	1·1268347	·20	1·7941226
·86	·5492229	·52	1·1436140	·19	1·8224880
·85	·5701209	·51	1·1604693	·18	1·8519171
·84	·5905140	·50	1·1774100	·17	1·8825285
·83	·6104582	·49	1·1944454	·16	1·9144615
·82	·6300015	·48	1·2115851	·15	1·9478809
·81	·6491857	·47	1·2288390	·14	1·9829840
·80	·6680472	·46	1·2462173	·13	2·0200103
·79	·6866183	·45	1·2637307	·12	2·0592540
·78	·7049275	·44	1·2813903	·11	2·1010830
·77	·7230004	·43	1·2992075	·10	2·1459660
·76	·7408601	·42	1·3171944	·09	2·1945139
·75	·7585276	·41	1·3353637	·08	2·2475447
·74	·7760220	·40	1·3537287	·07	2·3061917
·73	·7933609	·39	1·3723036	·06	2·3720922
·72	·8105604	·38	1·3911032	·05	2·4477468
·71	·8276356	·37	1·4101434	·04	2·5372725
·70	·8446004	·36	1·4294413	·03	2·6482288
·69	·8614681	·35	1·4490149	·02	2·7971496
·68	·8782511	·34	1·4688837	·01	3·0348543
·67	·8949610				

TABLE III.—Ordinates of curves of Divergence-diagram in terms of Abscissa.

Abscissa = ·25 ± θ or ·75 ± θ.

Ordinate = z.

$$z = e^{-\frac{1}{2}x^2 \sec^2 2\pi\theta}, \text{ the value of } x \text{ being given by } \alpha = \sqrt{\frac{2}{\pi}} \int_0^x e^{-\frac{1}{2}x^2} dx.$$

VALUES of x.

$\alpha = \cdot 0$; $x = 0\cdot 00000\ 00000$
$\alpha = \cdot 1$; $x = 0\cdot 12566\ 13469$
$\alpha = \cdot 2$; $x = 0\cdot 25334\ 71031$
$\alpha = \cdot 3$; $x = 0\cdot 38532\ 04664$
$\alpha = \cdot 4$; $x = 0\cdot 52440\ 05127$
$\alpha = \cdot 5$; $x = 0\cdot 67448\ 97502$
$\alpha = \cdot 6$; $x = 0\cdot 84162\ 12336$
$\alpha = \cdot 7$; $x = 1\cdot 03643\ 33895$
$\alpha = \cdot 8$; $x = 1\cdot 28155\ 15655$
$\alpha = \cdot 9$; $x = 1\cdot 64485\ 36270$

VALUES of z.

	$\alpha = \cdot 0$	$\alpha = \cdot 1$	$\alpha = \cdot 2$	$\alpha = \cdot 3$	$\alpha = \cdot 4$	$\alpha = \cdot 5$	$\alpha = \cdot 6$	$\alpha = \cdot 7$	$\alpha = \cdot 8$	$\alpha = \cdot 9$
θ	z	z	z	z	z	z	z	z	z	z
·00	1·00000	·99214	·96842	·92845	·87154	·79655	·70176	·58444	·43991	·25852
·01	1·00000	·99210	·96829	·92818	·87106	·79583	·70078	·58320	·43848	·25714
·02	1·00000	·99201	·96792	·92735	·86963	·79366	·69781	·57945	·43418	·25300
·03	1·00000	·99185	·96729	·92595	·86719	·78998	·69277	·57313	·42696	·24610
·04	1·00000	·99162	·96637	·92392	·86367	·78469	·68557	·56411	·41673	·23647
·05	1·00000	·99131	·96514	·92120	·85898	·77765	·67601	·55222	·40338	·22412
·06	1·00000	·99091	·96356	·91771	·85295	·76865	·66386	·53725	·38677	·20912
·07	1·00000	·99040	·96156	·91332	·84540	·75742	·64883	·51891	·36677	·19161
·08	1·00000	·98977	·95907	·90785	·83606	·74363	·63053	·49687	·34322	·17177
·09	1·00000	·98899	·95598	·90110	·82459	·72682	·60848	·47076	·31603	·14993
·10	1·00000	·98801	·95215	·89277	·81052	·70642	·58210	·44016	·28517	·12658
·11	1·00000	·98679	·94738	·88246	·79326	·68172	·55071	·40467	·25078	·10243
·12	1·00000	·98525	·94139	·86962	·77202	·65177	·51351	·36395	·21324	·07842
·13	1·00000	·98329	·93381	·85349	·74571	·61544	·46964	·31785	·17336	·05575
·14	1·00000	·98076	·92405	·83301	·71291	·57130	·41826	·26663	·13251	·03581
·15	1·00000	·97741	·91129	·80665	·67168	·51768	·35876	·21128	·09284	·01993
·16	1·00000	·97288	·89424	·77216	·61946	·45282	·29126	·15402	·05726	·00899
·17	1·00000	·96655	·87086	·72625	·55298	·37527	·21740	·09884	·02906	·00294
·18	1·00000	·95738	·83776	·66399	·46839	·28515	·14176	·05168	·01078	·00057
·19	1·00000	·94340	·78914	·57822	·36254	·18665	·07328	·01900	·00234	·00005
·20	1·00000	·92064	·71457	·45960	·23695	·09236	·02451	·00361	·00018	·00000
·21	1·00000	·88015	·59517	·30110	·10826	·02528	·00326	·00017	·00000	·00000
·22	1·00000	·79862	·40091	·12072	·01992	·00154	·00004	·00000	·00000	·00000
·23	1·00000	·60494	·12964	·00886	·00016	·00000	·00000	·00000	·00000	·00000
·24	1·00000	·13499	·00029	·00000	·00000	·00000	·00000	·00000	·00000	·00000
·25	Indeterminate	·00000	·00000	·00000	·00000	·00000	·00000	·00000	·00000	·00000

TABLE IV.—Abscissæ of curves of Divergence-diagram in terms of Ordinate.

(*Converse of Table III.*)

	$\alpha = \cdot 0$	$\alpha = \cdot 1$	$\alpha = \cdot 2$	$\alpha = \cdot 3$	$\alpha = \cdot 4$	$\alpha = \cdot 5$	$\alpha = \cdot 6$	$\alpha = \cdot 7$	$\alpha = \cdot 8$	$\alpha = \cdot 9$
z	θ	θ	θ	θ	θ	θ	θ	θ	θ	θ
1·00	Indeterminate	..	..	..	..	..	..	..	..	..
·99	·25000	·07662	..	..	..	..	..	..	..	..
·98	·25000	·14252	..	..	..	..	..	..	..	..
·97	·25000	·16498	..	..	..	..	..	..	..	..
·96	·25000	·17753	·07651	..	..	..	..	..	..	..
·95	·25000	·18583	·10478	..	..	..	..	..	..	..
·94	·25000	·19186	·12203	..	..	..	..	..	..	..
·93	·25000	·19650	·13422	..	..	..	..	..	..	..
·92	·25000	·20022	·14349	·05372	..	..	..	..	..	..
·91	·25000	·20328	·15087	·07632	..	..	..	..	..	..
·90	·25000	·20587	·15695	·09145	..	..	..	..	..	..
·89	·25000	·20809	·16207	·10291	..	..	..	..	..	..
·88	·25000	·21003	·16647	·11210	..	..	..	..	..	..
·87	·25000	·21174	·17031	·11973	·01795	..	..	..	..	..
·86	·25000	·21326	·17369	·12624	·04803	..	..	..	..	..
·85	·25000	·21463	·17671	·13189	·06417	..	..	..	..	..
·84	·25000	·21587	·17943	·13686	·07603	..	..	..	..	..
·83	·25000	·21700	·18189	·14128	·08553	..	..	..	..	..
·82	·25000	·21804	·18413	·14526	·09349	..	..	..	..	..
·81	·25000	·21900	·18619	·14886	·10033	..	..	..	..	..
·80	·25000	·21988	·18809	·15215	·10634	..	..	..	..	..
·79	·25000	·22071	·18985	·15517	·11168	·02996	..	..	..	..
·78	·25000	·22148	·19149	·15796	·11649	·04694	..	..	..	..
·77	·25000	·22220	·19302	·16054	·12085	·05863	..	..	..	..
·76	·25000	·22287	·19445	·16295	·12484	·06788	..	..	..	..
·75	·25000	·22351	·19580	·16520	·12851	·07563	..	..	..	..
·74	·25000	·22411	·19707	·16730	·13191	·08233	..	..	..	..
·73	·25000	·22468	·19827	·16929	·13507	·08825	..	..	..	..
·72	·25000	·22523	·19941	·17116	·13801	·09356	..	..	..	..
·71	·25000	·22574	·20049	·17293	·14079	·09838	..	..	..	..
·70	·25000	·22623	·20151	·17460	·14339	·10279	·01337	..	..	..
·69	·25000	·22670	·20249	·17620	·14584	·10686	·03423	..	..	..
·68	·25000	·22715	·20343	·17771	·14816	·11063	·04613	..	..	..
·67	·25000	·22758	·20432	·17916	·15036	·11415	·05523	..	..	..
·66	·25000	·22799	·20518	·18054	·15245	·11744	·06277	..	..	..
·65	·25000	·22839	·20600	·18187	·15445	·12054	·06929	..	..	..
·64	·25000	·22877	·20679	·18314	·15635	·12346	·07506	..	..	..
·63	·25000	·22914	·20755	·18436	·15817	·12622	·08026	..	..	..
·62	·25000	·22949	·20829	·18553	·15991	·12885	·08500	..	..	..
·61	·25000	·22983	·20899	·18666	·16158	·13134	·08936	..	..	..
·60	·25000	·23016	·20968	·18775	·16319	·13372	·09341	..	..	..
·59	·25000	·23048	·21034	·18881	·16473	·13600	·09718	..	..	..
·58	·25000	·23079	·21098	·18982	·16622	·13818	·10073	·01888	..	..
·57	·25000	·23109	·21160	·19081	·16766	·14027	·10406	·03383	..	..
·56	·25000	·23139	·21220	·19176	·16905	·14227	·10722	·04375	..	..
·55	·25000	·23167	·21279	·19269	·17040	·14421	·11021	·05163	..	..
·54	·25000	·23195	·21336	·19358	·17170	·14607	·11306	·05832	..	..
·53	·25000	·23221	·21391	·19446	·17296	·14787	·11577	·06419	..	..
·52	·25000	·23248	·21445	·19531	·17419	·14961	·11837	·06946	..	..

TABLE IV.—Abscissæ of curves of Divergence-diagram in terms of Ordinate (continued).

	α = ·0	α = ·1	α = ·2	α = ·3	α = ·4	α = ·5	α = ·6	α = ·7	α = ·8	α = ·9
z	θ	θ	θ	θ	θ	θ	θ	θ	θ	θ
·51	·25000	·23273	·21497	·19613	·17538	·15129	·12086	·07426	..	..
·50	·25000	·23298	·21548	·19694	·17653	·15292	·12326	·07868	..	..
·49	·25000	·23323	·21598	·19772	·17766	·15450	·12556	·08280	..	..
·48	·25000	·23346	·21647	·19849	·17876	·15603	·12778	·08664	..	..
·47	·25000	·23370	·21695	·19924	·17983	·15753	·12992	·09027	..	..
·46	·25000	·23392	·21742	·19997	·18088	·15898	·13200	·09369	..	..
·45	·25000	·23415	·21788	·20069	·18190	·16040	·13401	09695	..	..
·44	·25000	·23437	·21832	·20139	·18290	·16178	·13595	·10005	..	..
·43	·25000	·23458	·21876	·20208	·18387	·16312	·13785	·10301	·02627	..
·42	·25000	·23479	·21920	·20275	·18483	·16444	·13968	·10586	·03711	..
·41	·25000	·23500	·21962	·20341	·18577	·16573	·14147	·10859	·04534	..
·40	·25000	·23520	·22004	·20406	·18669	·16699	·14322	·11122	·05221	..
·39	·25000	·23541	·22045	·20470	·18760	·16822	·14492	·11376	·05821	..
·38	·25000	·23560	·22085	·20533	·18848	·16944	·14659	·11621	·06359	..
·37	·25000	·23580	·22125	·20595	·18936	·17062	·14821	·11859	·06850	..
·36	·25000	·23599	·22164	·20656	·19022	·17179	·14980	·12091	·07304	..
·35	·25000	·23618	·22203	·20716	·19106	·17294	·15137	·12315	·07727	..
·34	·25000	·23637	·22241	·20776	·19190	·17407	·15290	·12534	·08126	..
·33	·25000	·23655	·22279	·20834	·19272	·17518	·15440	·12747	·08503	..
·32	·25000	·23674	·22316	·20892	·19354	·17628	·15588	·12956	·08862	..
·31	·25000	·23692	·22353	·20949	·19434	·17736	·15733	·13160	·09205	..
·30	·25000	·23710	·22390	·21006	·19513	·17843	·15876	·13359	·09534	..
·29	·25000	·23728	·22426	·21062	·19592	·17949	·16018	·13555	·09851	..
·28	·25000	·23745	·22462	·21118	·19670	·18054	·16157	·13748	·10157	..
·27	·25000	·23763	·22498	·21174	·19748	·18157	·16295	·13937	·10453	..
·26	·25000	·23780	·22534	·21229	·19824	·18260	·16431	·14123	·10741	..
·25	·25000	·23798	·22569	·21283	·19901	·18362	·16566	·14307	·11022	·02485
·24	·25000	·23815	·22604	·21338	·19977	·18464	·16700	·14489	·11295	·03665
·23	·25000	·23832	·22640	·21392	·20053	·18565	·16833	·14668	·11563	·04550
·22	·25000	·23850	·22675	·21446	·20128	·18665	·16966	·14846	·11825	·05293
·21	·25000	·23867	·22710	·21501	·20204	·18766	·17098	·15022	·12083	·05946
·20	·25000	·23884	·22745	·21555	·20279	·18866	·17229	·15198	·12337	·06538
·19	·25000	·23902	·22780	·21609	·20355	·18966	·17360	·15372	·12588	·07085
·18	·25000	·23919	·22816	·21664	·20431	·19067	·17492	·15547	·12836	·07598
·17	·25000	·23937	·22852	·21719	·20507	·19168	·17623	·15721	·13082	·08084
·16	·25000	·23955	·22888	·21775	·20584	·19270	·17756	·15895	·13327	·08549
·15	·25000	·23973	·22924	·21831	·20662	·19372	·17889	·16070	·13572	·08997
·14	·25000	·23991	·22961	·21888	·20740	·19476	·18024	·16247	·13816	·09432
·13	·25000	·24009	·22999	·21945	·20820	·19582	·18160	·16425	·14062	·09857
·12	·25000	·24028	·23037	·22004	·20902	·19689	·18299	·16606	·14309	·10274
·11	·25000	·24048	·23076	·22065	·20985	·19799	·18441	·16790	·14560	·10688
·10	·25000	·24068	·23117	·22127	·21071	·19911	·18586	·16978	·14814	·11100
·09	·25000	·24088	·23159	·22191	·21160	·20028	·18736	·17171	·15075	·11514
·08	·25000	·24110	·23202	·22258	·21252	·20149	·18891	·17372	·15343	·11933
·07	·25000	·24132	·23248	·22328	·21349	·20276	·19054	·17582	·15623	·12361
·06	·25000	·24156	·23297	·22403	·21452	·20411	·19227	·17803	·15916	·12805
·05	·25000	·24183	·23350	·22484	·21564	·20557	·19414	·18041	·16230	·13272
·04	·25000	·24211	·23408	·22574	·21687	·20718	·19619	·18303	·16573	·13775
·03	·25000	·24245	·23475	·22676	·21827	·20901	·19853	·18600	·16960	·14334
·02	·25000	·24285	·23557	·22801	·21998	·21124	·20136	·18959	·17425	·14995
·01	·25000	·24341	·23670	·22974	·22236	·21433	·20528	·19453	·18062	·15884
·00	·25000	·25000	·25000	·25000	·25000	·25000	·25000	·25000	·25000	·25000

TABLE V.—Table for Calculation of Probable Error.

This table gives $Q\sqrt{N}$ in terms of N, where $Q = \cdot 67448975\ldots$, and N has any value. The values of N in the first column are the values corresponding to values of $Q\sqrt{N}$ intermediate between those in the second column. Thus $Q\sqrt{N} = 93\cdot 5$ gives $N = 19216$, and $Q\sqrt{N} = 94\cdot 5$ gives $N = 19630$; and therefore for any value of N between 19216 and 19630 the value of $Q\sqrt{N}$ to the nearest integer is 94. The figures in N are arranged in pairs, since the result of dividing $\sqrt{N}$ by 10 is to divide N by 100. Thus for $N = \cdot 01\ 93\ 00$ the value of $Q\sqrt{N}$ to three places of decimals is $\cdot 094$; and similarly, if $N = \cdot 00\ 00\ 01\ 93$, $Q\sqrt{N} = \cdot 00094$, correct to five places of decimals. Thus the table gives $Q\sqrt{N}$ within from $\cdot 8$ to $\cdot 08$ per cent. of its value, without the necessity for any interpolation. This is accurate enough for ordinary purposes.

N.	$Q\sqrt{N}$.	N.	$Q\sqrt{N}$.	N.	$Q\sqrt{N}$.
00 97 21		01 56 95		02 30 94	
	067		085		103
01 00 15		60 69		35 47	
	068		086		104
03 14		64 47		40 04	
	069		087		105
06 17		68 29		44 66	
	070		088		106
09 25		72 16		49 32	
	071		089		107
12 37		76 07		54 02	
	072		090		108
15 54		80 03		58 77	
	073		091		109
18 75		84 03		63 56	
	074		092		110
22 00		88 08		68 39	
	075		093		111
25 30		92 16		73 27	
	076		094		112
28 64		96 30		78 20	
	077		095		113
32 02		02 00 47		83 17	
	078		096		114
35 45		04 69		88 18	
	079		097		115
38 93		08 96		93 23	
	080		098		116
42 44		13 27		98 33	
	081		099		117
46 00		17 62		03 03 48	
	082		100		118
49 61		22 01		08 66	
	083		101		119
53 26		26 45		13 90	
	084		102		120

TABLE V.—Table for Calculation of Probable Error (continued).

N.	Q√N.	N.	Q√N.	N.	Q√N.
03 19 17	121	04 84 73	149	06 84 76	177
24 49	122	91 28	150	92 54	178
29 85	123	97 88	151	07 00 37	179
35 26	124	05 04 52	152	08 24	180
40 71	125	11 20	153	16 15	181
46 21	126	17 92	154	24 11	182
51 75	127	24 69	155	32 11	183
57 33	128	31 51	156	40 15	184
62 96	129	38 37	157	48 24	185
68 63	130	45 27	158	56 37	186
74 34	131	52 21	159	64 55	187
80 10	132	59 20	160	72 77	188
85 91	133	66 24	161	81 04	189
91 75	134	73 32	162	89 35	190
97 64	135	80 44	163	97 70	191
04 03 58	136	87 60	164	08 06 10	192
09 56	137	94 81	165	14 54	193
15 58	138	06 02 07	166	23 02	194
21 65	139	09 37	167	31 55	195
27 76	140	16 71	168	40 12	196
33 91	141	24 09	169	48 74	197
40 11	142	31 52	170	57 40	198
46 35	143	39 00	171	66 10	199
52 64	144	46 51	172	74 85	200
58 97	145	54 07	173	83 65	201
65 35	146	61 68	174	92 48	202
71 76	147	69 33	175	09 01 36	203
78 23	148	77 02	176	10 29	204

TABLE V.—Table for Calculation of Probable Error (continued).

N.	Q√N.	N.	Q√N.	N.	Q√N.
09 19 25		11 88 22		14 91 64	
	205		233		261
28 27		98 46		15 03 12	
	206		234		262
37 32		12 08 75		14 63	
	207		235		263
46 42		19 08		26 20	
	208		236		264
55 57		29 45		37 80	
	209		237		265
64 76		39 87		49 45	
	210		238		266
73 99		50 33		61 15	
	211		239		267
83 26		60 84		72 88	
	212		240		268
92 58		71 39		84 67	
	213		241		269
10 01 95		81 99		96 49	
	214		242		270
11 36		92 63		16 08 36	
	215		243		271
20 81		13 03 31		20 28	
	216		244		272
30 30		14 04		32 23	
	217		245		273
39 84		24 81		44 24	
	218		246		274
49 43		35 62		56 28	
	219		247		275
59 05		46 48		68 37	
	220		248		276
68 73		57 38		80 50	
	221		249		277
78 44		68 33		92 68	
	222		250		278
88 20		79 32		17 04 90	
	223		251		279
98 01		90 35		17 17	
	224		252		280
11 07 85		14 01 43		29 48	
	225		253		281
17 74		12 55		41 83	
	226		254		282
27 68		23 72		54 23	
	227		255		283
37 66		34 93		66 67	
	228		256		284
47 68		46 19		79 16	
	229		257		285
57 75		57 48		91 68	
	230		258		286
67 86		68 83		18 04 26	
	231		259		287
78 02		80 21		16 87	
	232		260		288

TABLE V.—Table for Calculation of Probable Error (continued).

N.	Q $\sqrt{N}$.	N.	Q $\sqrt{N}$.	N.	Q $\sqrt{N}$.
18 29 54	289	22 01 90	317	26 08 72	345
42 24	290	15 83	318	23 89	346
54 99	291	29 81	319	39 10	347
67 78	292	43 84	320	54 35	348
80 62	293	57 90	321	69 65	349
93 50	294	72 02	322	85 00	350
19 06 43	295	86 17	323	27 00 38	351
19 39	296	23 00 37	324	15 81	352
32 41	297	14 61	325	31 29	353
45 46	298	28 90	326	46 81	354
58 56	299	43 23	327	62 37	355
71 71	300	57 61	328	77 98	356
84 90	301	72 03	329	93 63	357
98 13	302	86 49	330	28 09 32	358
20 11 41	303	24 01 00	331	25 06	359
24 73	304	15 55	332	40 84	360
38 09	305	30 15	333	56 67	361
51 50	306	44 79	334	72 54	362
64 95	307	59 47	335	88 45	363
78 45	308	74 20	336	29 04 41	364
91 99	309	88 97	337	20 41	365
21 05 57	310	25 03 78	338	36 46	366
19 20	311	18 64	339	52 55	367
32 87	312	33 55	340	68 68	368
46 59	313	48 49	341	84 86	369
60 35	314	63 48	342	30 01 08	370
74 16	315	78 52	343	17 35	371
88 00	316	93 60	344	33 66	372

TABLE V.—Table for Calculation of Probable Error (continued).

N.	Q√N.	N.	Q√N.	N.	Q√N.
30 50 01	373	35 25 77	401	40 36 00	429
66 41	374	43 40	402	54 86	430
82 85	375	61 07	403	73 76	431
99 34	376	78 79	404	92 71	432
31 15 87	377	96 55	405	41 11 70	433
32 44	378	36 14 36	406	30 74	434
49 06	379	32 21	407	49 82	435
65 72	380	50 10	408	68 94	436
82 43	381	68 03	409	88 11	437
99 18	382	86 02	410	42 07 32	438
32 15 97	383	37 04 04	411	26 57	439
32 81	384	22 11	412	45 87	440
49 69	385	40 22	413	65 22	441
66 62	386	58 38	414	84 60	442
83 59	387	76 58	415	43 04 04	443
33 00 60	388	94 82	416	23 51	444
17 66	389	38 13 11	417	43 03	445
34 76	390	31 44	418	62 59	446
51 90	391	49 82	419	82 20	447
69 09	392	68 24	420	44 01 85	448
86 32	393	86 70	421	21 55	449
34 03 60	394	39 05 21	422	41 29	450
20 92	395	23 76	423	61 07	451
38 29	396	42 36	424	80 90	452
55 70	397	61 00	425	45 00 77	453
73 15	398	79 68	426	20 68	454
90 65	399	98 41	427	40 64	455
35 08 19	400	40 17 18	428	60 64	456

TABLE V.—Table for Calculation of Probable Error (continued).

N.	$Q\sqrt{N}$.	N.	$Q\sqrt{N}$.	N.	$Q\sqrt{N}$.
45 80 69		51 59 85		57 73 47	
	457		485		513
46 00 78		81 17		96 02	
	458		486		514
20 91		52 02 53		58 18 62	
	459		487		515
41 09		23 94		41 26	
	460		488		516
61 32		45 40		63 95	
	461		489		517
81 58		66 90		86 67	
	462		490		518
47 01 89		88 44		59 09 45	
	463		491		519
22 25		53 10 02		32 26	
	464		492		520
42 65		31 65		55 12	
	465		493		521
63 09		53 32		78 03	
	466		494		522
83 57		75 04		60 00 98	
	467		495		523
48 04 11		96 80		23 97	
	468		496		524
24 68		54 18 61		47 00	
	469		497		525
45 30		40 46		70 08	
	470		498		526
65 96		62 35		93 21	
	471		499		527
86 67		84 29		61 16 38	
	472		500		528
49 07 42		55 06 27		39 59	
	473		501		529
28 21		28 29		62 84	
	474		502		530
49 05		50 36		86 14	
	475		503		531
69 93		72 48		62 09 49	
	476		504		532
90 86		94 63		32 88	
	477		505		533
50 11 83		56 16 83		56 31	
	478		506		534
32 84		39 08		79 78	
	479		507		535
53 90		61 37		63 03 30	
	480		508		536
75 00		83 70		26 87	
	481		509		537
96 15		57 06 08		50 48	
	482		510		538
51 17 34		28 50		74 13	
	483		511		539
38 57		50 96		97 82	
	484		512		540

TABLE V.—Table for Calculation of Probable Error (continued).

N.	$Q\sqrt{N}$.	N.	$Q\sqrt{N}$.	N.	$Q\sqrt{N}$.
64 21 56	541	71 04 12	569	78 21 14	597
45 35	542	29 13	570	47 39	598
69 17	543	54 19	571	73 68	599
93 04	544	79 29	572	79 00 01	600
65 16 96	545	72 04 44	573	26 39	601
40 92	546	29 63	574	52 81	602
64 92	547	54 87	575	79 27	603
88 97	548	80 14	576	80 05 78	604
66 13 06	549	73 05 47	577	32 34	605
37 20	550	30 83	578	58 93	606
61 38	551	56 24	579	85 57	607
85 60	552	81 70	580	81 12 26	608
67 09 87	553	74 07 19	581	38 99	609
34 18	554	32 74	582	65 76	610
58 53	555	58 32	583	92 58	611
82 93	556	83 95	584	82 19 44	612
68 07 37	557	75 09 63	585	46 34	613
31 86	558	35 34	586	73 29	614
56 39	559	61 11	587	83 00 29	615
80 97	560	86 91	588	27 32	616
69 05 59	561	76 12 76	589	54 40	617
30 25	562	38 66	590	81 53	618
54 96	563	64 59	591	84 08 70	619
79 71	564	90 57	592	35 91	620
70 04 50	565	77 16 60	593	63 17	621
29 34	566	42 67	594	90 47	622
54 22	567	68 78	595	85 17 81	623
79 15	568	94 94	596	45 20	624

TABLE V.—Table for Calculation of Probable Error (continued).

N.	Q√N.	N.	Q√N.	N.	Q√N.
85 72 63	625	90 45 71	642	95 31 49	659
86 00 11	626	73 93	643	60 46	660
27 63	627	91 02 20	644	89 48	661
55 19	628	30 51	645	96 18 54	662
82 80	629	58 87	646	47 64	663
87 10 45	630	87 27	647	76 79	664
38 15	631	92 15 71	648	97 05 98	665
65 89	632	44 20	649	35 21	666
93 67	633	72 73	650	64 49	667
88 21 50	634	93 01 31	651	93 81	668
49 37	635	29 92	652	98 23 18	669
77 29	636	58 59	653	52 59	670
89 05 25	637	87 30	654	82 05	671
33 25	638	94 16 05	655	99 11 54	672
61 30	639	44 84	656	41 09	673
89 39	640	73 68	657	70 67	674
90 17 53	641	95 02 56	658	100 00 30	

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

TABLE VI.—Abscissa of Standard Normal Curve in terms of Class-Index.

Class-Index = α.

Abscissa = x.

$$\alpha = \sqrt{\frac{2}{\pi}} \int_0^x e^{-\frac{1}{2}x^2} dx.$$

α.	x.	α.	x.	α.	x.
·00	·00000	·34	·43991	·67	·97411
·01	·01253	·35	·45376	·68	·99446
·02	·02507	·36	·46770	·69	1·01522
·03	·03761	·37	·48173	·70	1·03643
·04	·05015	·38	·49585	·71	1·05812
·05	·06271	·39	·51007	·72	1·08032
·06	·07527	·40	·52440	·73	1·10306
·07	·08784	·41	·53884	·74	1·12639
·08	·10043	·42	·55338	·75	1·15035
·09	·11304	·43	·56805	·76	1·17499
·10	·12566	·44	·58284	·77	1·20036
·11	·13830	·45	·59776	·78	1·22653
·12	·15097	·46	·61281	·79	1·25357
·13	·16366	·47	·62801	·80	1·28155
·14	·17637	·48	·64335	·81	1·31058
·15	·18912	·49	·65884	·82	1·34076
·16	·20189	·50	·67449	·83	1·37220
·17	·21470	·51	·69031	·84	1·40507
·18	·22754	·52	·70630	·85	1·43953
·19	·24043	·53	·72248	·86	1·47579
·20	·25335	·54	·73885	·87	1·51410
·21	·26631	·55	·75542	·88	1·55477
·22	·27932	·56	·77219	·89	1·59819
·23	·29237	·57	·78919	·90	1·64485
·24	·30548	·58	·80642	·91	1·69540
·25	·31864	·59	·82389	·92	1·75069
·26	·33185	·60	·84162	·93	1·81191
·27	·34513	·61	·85962	·94	1·88079
·28	·35846	·62	·87790	·95	1·95996
·29	·37186	·63	·89647	·96	2·05375
·30	·38532	·64	·91537	·97	2·17009
·31	·39886	·65	·93459	·98	2·32635
·32	·41246	·66	·95417	·99	2·57583
·33	·42615				

ERRATA.

P. 123, line 9. *For* $f'(\alpha', \beta', \gamma', \ldots)$, *read* $f(\alpha', \beta', \gamma', \ldots)$.

P. 128. The expression at the end of § 20 should be multiplied by k^2.

P. 131, line 8. *For* $2a^2/n$, *read* $2a^4/n$.